TD : Matrices stochastiques

1) a) A, B stochastiques

$$AB\Omega = A\Omega = \Omega$$

De plus, les coefficients de AB sont positifs.
AB est stochastique.

b) Par récurrence immédiate.

2) a) $$\left\| A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_\infty = \sup_{1\leq i\leq n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \leq \sup_{1\leq i\leq n} \sum_{j=1}^{n} a_{ij} |x_j|$$

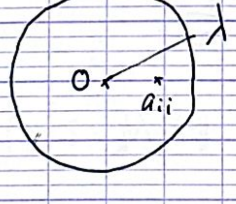
$$\leq \sum_{j=1}^{n} a_{ij} \|X\|_\infty = \|X\|_\infty$$

$$A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad |||A|||_\infty = 1$$

Si $\begin{cases} AX = \lambda X \\ X \neq 0 \end{cases}$ $\quad |\lambda| = \dfrac{\|AX\|}{\|X\|} \leq 1$

Si $a_{ii} > 0$ pour tout $1 \leq i \leq n$.
Supposons que $A - \lambda I$ non inversible

$$\underset{\text{Hadamard}}{\Longrightarrow} \exists i, \quad |a_{ii} - \lambda| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} = 1 - a_{ii}$$



Si $|\lambda| = 1$, $\lambda \neq 1$, $|a_{ii} - \lambda| > 1 - a_{ii}$

Sinon, $|\lambda| = 1 = |\lambda - a_{ii} + a_{ii}|$

$$\leq |\lambda - a_{ii}| + a_{ii} \leq 1 - a_{ii} + a_{ii} = 1$$

Égalité de Minkowski :

O, a_{ii}, $\lambda - a_{ii}$ alignés.

$$\lambda = 1$$

<u>Conséquences</u> : $\forall i$, $a_{ii} > 0$, la suite A^p converge

$$u = f_A$$

Soit $\lambda \in Spec(u)$

1) $\lambda \neq 1$, $|\lambda| < 1$. Sur F_λ : $u|_{F_\lambda} = \lambda I + \nu$

avec ν nilpotent.

$v = u|_{F_\lambda}$, $\quad v^p = (\lambda I + \nu)^p \underset{p \geq n}{=} \sum_{k=0}^{n-1} \binom{p}{k} \lambda^{p-k} \nu^k$

$$\underbrace{|||\nu^p|||}_{\to 0} \leq |\lambda|^{p-n} \times \underbrace{\sum_{k=0}^{n-1} \binom{p}{k}}_{\text{polynôme en } p} A \quad \text{avec } A \text{ majore les } |||\nu^k|||$$

$\lambda = 1 = |||A|||$

$u|_{E_1} = Id$

Sinon, $v = I + \nu$, $\nu \neq 0$

Soit $X \in Ker\, \nu^{d-2}$, $\nu(X) \neq 0$, $\nu^2(X) = 0$

$$\|v^p(X)\| = \|X + p\,\nu(X)\| \underset{p \to +\infty}{\longrightarrow} +\infty \quad ⚡$$

3. a) $\mathbb{P}(N_{k+1} = i) = \mathbb{P}(N_{k+1} = i \mid N_k = i-1) \times \mathbb{P}(N_k = i-1) + \mathbb{P}(N_{k+1} = i \mid N_k = i) \times \mathbb{P}(N_k = i)$

$+ \mathbb{P}(N_{k+1} = i \mid N_k = i+1) \times \mathbb{P}(N_k = i+1)$

$= \mathbb{P}(N_k = i-1) \times \dfrac{M-(i-1)}{M} \times \dfrac{1}{2} + \mathbb{P}(N_k = i) \times \dfrac{1}{2} +$

$\mathbb{P}(N_k = i+1) \times \dfrac{i+1}{M} \times \dfrac{1}{2}$

$$\mathbb{P}(N_{k+1} = i) = \frac{M-(i-1)}{2M} \mathbb{P}(N_k = i-1) + \frac{i+1}{2M} \mathbb{P}(N_k = i+1) + \frac{1}{2} \mathbb{P}(N_k = i)$$

b) $\mathbb{E}(N_{k+1}) = (0 \ 1 \ 2 \ \dots \ M)\, X_{k+1}$

$= (0 \ 1 \ 2 \ \dots \ M)\, B X_k$

$$= (0\ 1\ 2 \cdots M)\left(\frac{1}{2}I + \begin{pmatrix} 0 & \frac{1}{2M} & 0 & 0 & \cdots \\ \frac{M}{2M} & 0 & \frac{2}{2M} & & \\ & \frac{M-1}{2M} & 0 & & \\ & & & \ddots & \\ & & & 0 & \frac{M}{2M} \\ & & & \frac{1}{2M} & 0 \end{pmatrix}\right) X_k$$

$$= \frac{1}{2}(0\ 1 \cdots M) X_k + (0\ 1\ 2 \cdots M)\begin{pmatrix} \\ \\ \end{pmatrix} X_k$$

$$E(N_{k+1}) = \sum_{i=0}^{M} i\,\mathbb{P}(X_{k+1}=i)$$

$$= \frac{1}{2}\sum_{i=0}^{M} i\,\mathbb{P}(X_k=i) + \frac{1}{2M}\left[\sum_{i=1}^{M}(M-(i-1))\,\mathbb{P}(N_k=i-1) + \sum_{i=0}^{M-1}(i+1)\,\mathbb{P}(N_k=i+1)\right]$$

$=$

$$= \frac{E(N_k)}{2} + \frac{M-2}{2M}E(N_k) + \frac{1}{2} = \frac{1}{2} + \left(1-\frac{1}{M}\right)E(N_k)$$

$$l = \frac{1}{2} + \left(1-\frac{1}{M}\right) l \Rightarrow l = \frac{M}{2}$$

$$\Rightarrow E(N_k) - \frac{M}{2} = \underbrace{\left(1-\frac{1}{M}\right)^k}_{\longrightarrow 0}\left(E(N_0) - \frac{M}{2}\right)$$

Donc, $E(N_k) \longrightarrow \frac{M}{2}$

c)

$$A = \begin{pmatrix} a_{11} & & * \\ a_{21} & \ddots & \\ 0 & a_{n\,n-1} & a_{nn} \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} a_{11}-\lambda & & * \\ a_{21} & \ddots & \\ & a_{n\,n-1} & a_{nn}-\lambda \end{pmatrix}$$

$\neq 0$

$a_{21} \neq 0, \ldots, a_{n\,(n-1)} \neq 0$ mineur $\neq 0$

$rg(A-\lambda I) \geq n-1$, si $\lambda \in Spec(A)$, $rg(A-\lambda I) = n-1$

$\dim E_{\lambda, A} = 1$

$$\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix} A \begin{pmatrix}\frac{1}{\alpha_1} & & & \\ & \frac{1}{\alpha_2} & & \\ & & \ddots & \\ & & & \frac{1}{\alpha_n}\end{pmatrix}$$

B est à spectre réel et simple

$$D\,A\,D^{-1} = \left[\frac{\alpha_i a_{ij}}{\alpha_j}\right]_{1\leq i,j\leq n}$$

$$\begin{pmatrix}\alpha_1 & & \\ & \alpha_2 & \\ & & \alpha_3\end{pmatrix}\begin{pmatrix}0 & a_2 & 0 \\ a_1 & 0 & a_1 \\ 0 & a_2 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\alpha_1} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & \frac{1}{\alpha_3}\end{pmatrix}$$

$$= \begin{pmatrix}0 & \frac{\alpha_1}{\alpha_2}a_2 & 0 \\ \frac{\alpha_2}{\alpha_1}a_1 & 0 & \frac{\alpha_2}{\alpha_3}a_1 \\ 0 & \frac{\alpha_3}{\alpha_2}a_2 & 0\end{pmatrix}$$

$$\frac{\alpha_2}{\alpha_1}a_1 = \frac{\alpha_1}{\alpha_2}a_2 \ (?)$$

$$\left(\frac{\alpha_2}{\alpha_1}\right)^2 = \frac{a_2}{a_1} \qquad \frac{\alpha_2}{\alpha_1} = \sqrt{\frac{a_2}{a_1}}$$

$$A = \begin{pmatrix}0 & a_{n-1} & & & 0 \\ a_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_1 \\ 0 & & & a_{n-1} & 0\end{pmatrix}$$

$$\frac{\alpha_1}{\alpha_2}a_1 = \frac{\alpha_2}{\alpha_1}a_{n-1}$$

$$\frac{\alpha_1}{\alpha_2} = \sqrt{\frac{a_{n-1}}{a_1}} \quad \frac{\alpha_2}{\alpha_3} = \sqrt{\frac{a_{n-2}}{a_2}} \ \ldots$$

‖ A est semblable à $A' \in S_n(\mathbb{R})$, elle est donc $\mathbb{R}$-diagonalisable.

(cc) $${}^tB = P\begin{pmatrix}1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n\end{pmatrix}P^{-1}$$ avec $|\lambda_i| < 1$ pour $i \in [\![2, n]\!]$

$${}^tB^p \xrightarrow[p\to+\infty]{} \begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots \\ & & & 0\end{pmatrix}$$

$$X_{p+1} = {}^tB^p X_1 \to P\begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots \ 0\end{pmatrix}P^{-1}X_1$$

$$\sum_{i=0}^{M} z_i = 2^{-M}\sum_{i=0}^{M}\binom{M}{i} = 2^{-M}(1+1)^M = 1, \ Z \in \Delta$$

$$[BZ]_{i1} = \sum_{k=0}^{M} b_{ik}z_k = \begin{cases} b_{00}z_0 + b_{01}z_1 \quad i=0 \\ = \frac{1}{2}\times 2^{-M} + \frac{1}{2M}\times M 2^{-M} = 2^{-M} \\ b_{i\,i-1}z_{i-1} + \frac{1}{2}z_i \quad i \in [\![1, M-1]\!] \\ + b_{i\,i+1}z_{i+1} \\ = \frac{M-i}{2M}\times 2^{-M}\binom{M}{i-1} + \frac{1}{2}2^{-M}\binom{M}{i} \\ + \frac{i+1}{2M}\times 2^{-M}\binom{M}{i+1}\end{cases}$$

$$= 2^{-M} \times \frac{1}{2M}\left((M-i)\binom{M}{i-1} + M\binom{M}{i} + (i+1)\binom{M}{i+1}\right)$$

$$= 2^{-M} \times \frac{1}{2M}\left(\frac{M!\,(M-i)}{(i-1)!\,(M-i+1)!} + \frac{M \cdot M!}{i!\,(M-i)!} + \frac{M!}{i!\,(M-i-1)!}\right)$$

$$= 2^{-M} \times \frac{1}{2M}\left(\frac{M!\,(M-i)}{(i-1)!\,(M-i+1)!} + \frac{M \cdot M! + M!\,(M-i)}{i!\,(M-i)!}\right)$$

$$= 2^{-M} \times \frac{1}{2M} \times \frac{M!}{i!\,(M-i)!}\underbrace{\left(\frac{(M-i)\,i}{M-i+1} + M + M - i\right)}_{\frac{Mi - i^2 + 2M^3 - 2Mi + 2M - Mi + i^2 -}{M-i+1}}$$

$$= \frac{2M^3 - 2Mi + 2M - i}{M-i+1}$$

$$=$$

b)
$$\mathbb{E}(N_{k+1}) = \sum_{i=0}^{M} i\,\mathbb{P}(N_{k+1}=i)$$
$$= \frac{1}{2}\sum_{i=0}^{M} i\,\mathbb{P}(N_k=i) + \frac{1}{2M}\left(\sum_{i=0}^{M} i\,(i+1)\,\mathbb{P}(N_k=i+1)\right.$$
$$\left. + \sum_{i=0}^{M} i\,(M-(i-1))\,\mathbb{P}(N_k=i-1)\right)$$
$$= \frac{1}{2}\mathbb{E}(N_k) + \frac{1}{2M}\left(\sum_{i=1}^{M}(i-1)\,i\,\mathbb{P}(N_k=i) + \sum_{i=0}^{M-1}(i+1)(M-i)\mathbb{P}(N_k=\right.$$
$$= \frac{1}{2}\mathbb{E}(N_k) + \frac{1}{2M}\left(\sum_{i=0}^{M}(i^2-i+Mi+M-i^2-i)\,\mathbb{P}(N_k=i)\right)$$
$$= \frac{1}{2}\mathbb{E}(N_k) + \frac{1}{2M}\left(\sum_{i=0}^{M}(M-2)i\,\mathbb{P}(N_k=i) + (M+1)\right)$$
$$= \frac{1}{2}\mathbb{E}(N_k) + \frac{1}{2} + \frac{M-2}{2M}\mathbb{E}(N_k)$$
$$= \left(1-\frac{1}{M}\right)\mathbb{E}(N_k) + \frac{1}{2}$$

$$\mathbb{E}(N_{k+1}) - \frac{M}{2} = \left(1-\frac{1}{M}\right)\left(\mathbb{E}(N_k)\right) + \frac{M}{2}\left(\frac{1}{M}-1\right)$$
$$= \left(1-\frac{1}{M}\right)\left(\mathbb{E}(N_k)-\frac{M}{2}\right)$$
$$\mathbb{E}(N_k) = \underbrace{\left(1-\frac{1}{M}\right)^k\left(\mathbb{E}(N_0)-\frac{M}{2}\right)}_{\xrightarrow[k\to+\infty]{} 0} + \frac{M}{2}$$

Donc, $\lim\limits_{k\to+\infty}\mathbb{E}(N_k) = \dfrac{M}{2}$

c)
$$B - I = \begin{pmatrix} -\frac{1}{2} & * & & & \\ * & -\frac{1}{2} & * & & 0 \\ & * & \ddots & \ddots & \\ & & \ddots & \ddots & * \\ 0 & & & * & -\frac{1}{2} \end{pmatrix}$$

mineur $\neq 0$

$rg(B-I) \geqslant M$

$\dim E_{B,1} = 1$ ou 0

On calcule :

pour z'_0, $z'_0 = \frac{1}{2} z_0 + \frac{1}{2M} z_1$

$= \frac{1}{2} 2^{-M} + \frac{1}{2M} 2^{-M} \binom{M}{1} = 2^{-M}$

pour z'_M, $z'_M = \frac{1}{2} z_M + \frac{1}{2M} z_{M-1}$

$= 2^{-M}\left(\frac{1}{2} + \frac{1}{2M} \times M\right) = 2^{-M}$

pour $i \in [\![1, M-1]\!]$,

$$z'_i = \frac{1}{2} z_i + \frac{M-(i-1)}{2M} z_{i-1} + \frac{i+1}{2M} z_{i+1}$$

$$= \frac{1}{2^M}\left(\frac{1}{2}\binom{M}{i} + \frac{M-(i-1)}{2M}\binom{M}{i-1} + \frac{i+1}{2M}\binom{M}{i+1}\right)$$

$$= \frac{1}{2^M}\left(\frac{1}{2}\binom{M}{i} + \frac{M-(i-1)}{2M} \frac{M!}{(i-1)!(M-i+1)!} + \frac{i+1}{2M}\frac{M!}{(i+1)!(M-i-1)!}\right)$$

$$= \frac{1}{2^M}\left(\frac{1}{2}\binom{M}{i} + \frac{1}{2M}\left(\frac{M!}{(i-1)!(M-i)!} + \frac{M!}{i!(M-i-1)!}\right)\right)$$

$$= \frac{1}{2^M}\left(\frac{1}{2}\binom{M}{i} + \frac{1}{2}\left(\binom{M-1}{i-1} + \binom{M-1}{i}\right)\right)$$

$$= 2^{-M}\left(\frac{1}{2}\binom{M}{i} + \frac{1}{2}\binom{M}{i}\right) = 2^{-M}\binom{M}{i} = z_i$$

On a vérifié que $BZ = Z$.
Donc, $\mathrm{Vect}(Z) = E_{B,1}$, concernant la norme,
$E_{B,1} \cap \Delta = \{Z\}$.

d)

$X_{k+1} - Z = BX_k - Z = B(X_k - Z)$

$X_k - Z = B^k (X_0 - Z)$

$B = P \begin{pmatrix} 1 & & \\ & \lambda_1 & 0 \\ 0 & & \ddots \\ & & & \lambda_M \end{pmatrix} P^{-1}$, $|\lambda_1|, \ldots, |\lambda_M| < 1$,

$$\lim_{k\to+\infty} B^k = P\begin{pmatrix}1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0\end{pmatrix}P^{-1}$$

D'où, pour un X_0 donné, $(B^k X_0)$ est une suite convergente, mettons vers X.

$$X = BX \Rightarrow X = Z.$$

Donc, pour toute X_0, on a : $\lim_{k\to+\infty} X_k = Z$.

$$\mathbb{E}(N_k) \xrightarrow[k\to+\infty]{} \mathbb{E}(Z) = \sum_{i=0}^{M} i\, 2^{-M}\binom{M}{i}$$

$$= 2^{-M}\sum_{i=1}^{M}\frac{M!}{(i-1)!(M-i)!}$$

$$= 2^{-M} M \sum_{i=0}^{M-1}\frac{(M-1)!}{i!(M-1-i)!}$$

$$= 2^{-M} M \sum_{i=0}^{M-1}\binom{M-1}{i} = 2^{-M} M\cdot 2^{M-1} = \frac{M}{2}$$

$$\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix} A \begin{pmatrix}\frac{1}{\alpha_1} & & \\ & \ddots & \\ & & \frac{1}{\alpha_n}\end{pmatrix}$$

$$= \begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_n\end{pmatrix}\begin{pmatrix}\frac{a_{11}}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \cdots & \frac{a_{1n}}{\alpha_n} \\ \vdots & \vdots & & \vdots \\ \frac{a_{n1}}{\alpha_1} & \frac{a_{n2}}{\alpha_n} & & \frac{a_{nn}}{\alpha_n}\end{pmatrix} = \begin{pmatrix}a_{11} & \frac{\alpha_1}{\alpha_2}a_{12} & \cdots & \frac{\alpha_1}{\alpha_n}a_{1n} \\ \frac{\alpha_2}{\alpha_1}a_{21} & a_{22} & & \\ \vdots & & \ddots & \\ \frac{\alpha_n}{\alpha_1}a_{n1} & & & a_{nn}\end{pmatrix}$$

$$= \left[\frac{\alpha_i}{\alpha_j}a_{ij}\right]_{1\leq i,j\leq n}$$