TD : Séries entières

1.1

Soit $M > 0$ t.q. $\forall n \in \mathbb{N}$, $|u_n| \leq M$.

Soit $z \in \mathbb{C}$ t.q. $|z| \leq 1$

$\forall n \in \mathbb{N}$, $|u_n z^n| = |u_n| |z|^n \leq M$

Donc, $\rho(\sum u_n z^n) \geq 1$.

$u_n \not\to 0$ sinon, $u_n + \frac{u_{3n}}{3} \to 0$ ↯.

Donc, $\sum u_n \times 1^n$ ne CV pas.

$\rho(\sum u_n z^n) \leq 1$. Ainsi, $\rho(\sum u_n z^n) = 1$.

1.2.

* $\frac{a_{n+1}}{a_n} = (2+\sqrt{3})$ Donc, $\rho(\sum (2+\sqrt{3})^n z^n) = 2-\sqrt{3}$

** $(2+\sqrt{3})^n + (2-\sqrt{3})^n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \sqrt{3}^k + \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-\sqrt{3})^k$

pour k impair, $\binom{n}{k} 2^{n-k} \sqrt{3}^k + \binom{n}{k} 2^{n-k} (-\sqrt{3})^k = 0$

Donc, $(2+\sqrt{3})^n + (2-\sqrt{3})^n \in \mathbb{Z}$ et $(2-\sqrt{3}) < 0,5$

donc, pour $n \geq 1$, $a_n = (2-\sqrt{3})^n$.

$\frac{a_{n+1}}{a_n} = 2-\sqrt{3}$ D'où, $\rho(\sum a_n z^n) = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}$

1.3

Clairement, $\rho(\sum a_n z^n) \leq 1$ car $\sum a_n$ DV.

$\frac{a_n}{A_n} = \frac{A_n - A_{n-1}}{A_n} \to 0 \Rightarrow \frac{A_{n-1}}{A_n} \to 1$

Donc, $\rho(\sum A_n z^n) = 1$.

Soit $z \in]0, 1[$.

$$\underbrace{\sum_0^N A_n z^n (1-z)}_{\text{CV lorsque } N \to +\infty} = \sum_0^N A_n z^n - \sum_0^N A_n z^{n+1}$$

$$= \underbrace{A_0}_{\text{cste}} + \sum_{n=1}^{N} \underbrace{(A_n - A_{n-1})}_{a_n} z^n - \underbrace{A_N z^{N+1}}_{\to 0}$$

Donc, $\rho(\sum a_n z^n) = 1$.

1.4 * $\left|\frac{z^n}{n\sin(n\pi\sqrt{3})}\right| \geq \frac{|z|^n}{n}$ si $|z| \geq 1$, $\sum$ DV

Donc, $\rho\left(\sum \frac{z^n}{n\sin(n\pi\sqrt{3})}\right) \leq 1$.

** On choisit $m \in \mathbb{Z}$ t.q. $|n\pi\sqrt{3} - m\pi| < \frac{\pi}{2}$

$$\Rightarrow |\sin(n\pi\sqrt{3} - m\pi)| \geq \frac{2}{\pi}|n\pi\sqrt{3} - m\pi|$$

$$= 2n\left|\sqrt{3} - \frac{m}{n}\right|$$

$$= \frac{2n}{n}\left|\frac{(\sqrt{3}n - m)(\sqrt{3}n+m)}{\sqrt{3}n+m}\right|$$

$$= 2\left|\frac{3n^2 - m^2}{\sqrt{3}n+m}\right| \geq \frac{2C}{n} \quad C = c^{te}.$$

Donc, $\left|\frac{z^n}{n\sin(n\pi\sqrt{3})}\right| \leq \frac{|z|^n}{2C}$ qui est borné pour $|z|<1$.

Donc, $\rho\left(\sum \frac{z^n}{n\sin(n\pi\sqrt{3})}\right) = 1$.

2.1 $\forall x \in \mathbb{R}_+$, $e^x - P(x) > 0$.

$e^x - P(x) \to +\infty$ donc $M = \inf_{x\in\mathbb{R}_+}(e^x - P(x))$

atteint en $x_0 \in \mathbb{R}_+$.

$\forall M$ $\exists m$ ‖ Sur $[0, M]$, $Q_m(x) > e^x - \frac{M}{2}$

On fixe $N > \deg P$, $Q_N(x) - P(x) \xrightarrow[x\to+\infty]{} +\infty$

$\exists A \in \mathbb{R}$ $\forall x \geq A$, $Q_N(x) - P(x) \geq M$

Sur $[0, A]$, il existe $m \geq N$ t.q. $Q_m(x)$

$\geq e^x - \frac{M}{2} > P(x)$

Mais pour $x \geq A$. $Q_m(x) \geq Q_N(x) > P(x)$.

Donc, $Q_m(x) > P(x)$ sur $\mathbb{R}_+$.

2.2

Pour $z=1$, $\sum |a_n| \leqslant |a_1| + \sum_{n=2}^{+\infty} |a_n| \leqslant |a_1| + |a_1|$

Donc, $\rho(\sum a_n z^n) \geqslant 1$.

Soit $z_1, z_2 \in D(0,1)$ t.q. $f(z_1) = f(z_2)$

On suppose que $z_1 \neq z_2$,

$$-a_1(z_1 - z_2) = \sum_{n=2}^{+\infty} a_n (z_1^n - z_2^n)$$

$$a_1 = -\sum_{n=2}^{+\infty} a_n \underbrace{(z_1^{n-1} + z_1^{n-2} z_2 + \dots + z_2^{n-1})}_{|''| < n}$$

Donc, $|a_1| < \sum_{n=2}^{+\infty} n |a_n|$, ABS ! $z_1 = z_2$ f est injective

2.3

a) Par passage à la limite, $|b_k| \leqslant \alpha_k$ sommable.

Soit $\varepsilon > 0$. $N \in \mathbb{N}$, $n > N$.

$$\left|\sum_{k=0}^{+\infty} a_{k,n} - b_k\right| \leqslant \sum_{k=0}^{N} |a_{k,n} - b_k| + \sum_{k>N+1}^{+\infty} |a_{k,n} - b_k|$$

$$\leqslant \sum_{k=0}^{N} |a_{k,n} - b_k| + 2\sum_{k=N+1}^{+\infty} \alpha_k$$

On suppose N t.q. $\sum_{k=N+1}^{+\infty} \alpha_k \leqslant \varepsilon$.

$n_0 \in \mathbb{N}$, $n \geqslant n_0 \Rightarrow \forall k \in [\![0, N]\!]$, $|b_k - a_{k,n}| \leqslant \frac{\varepsilon}{2^k}$

$$\left|\sum_{k=0}^{+\infty} a_{k,n} - b_k\right| \leqslant 2\varepsilon + 2\varepsilon = 4\varepsilon$$

b) $u_{n,p} = \dfrac{a_p b_{n-p}}{b_n}$ si $p \leqslant 0$, $p > n$, 0

$$u_{n,p} \xrightarrow[n \to +\infty]{} a_p \beta^p$$

On veut $|u_{n,p}| \leqslant \alpha_p$ $\sum \alpha_p < +\infty$

Soient r : $|\beta| < r < 1$ et N t.q. $\forall k \geqslant N$, $\left|\frac{b_{k-1}}{b_k}\right| < r$

Si $n - p \geqslant N$, $\left|\frac{b_{n-p}}{b_n}\right| \leqslant r^p$. $|u_{n,p}| \leqslant r^p$

$n-p < N$ fini

$n-p \in \{0, \dots, N-1\}$

$$\left|\frac{b_0}{b_n}\right| = \left|\frac{b_0}{b_N}\underbrace{\frac{b_N}{b_n}}_{\leq 1}\right| \leq \left|\frac{b_0}{b_N}\right|\gamma^{n-N}$$

$$\left|\frac{b_1}{b_n}\right| \leq \left|\frac{b_1}{b_N}\right|\gamma^{n-N} \qquad \left|a_n \cdot \frac{b_0}{b_n}\right| \leq |a_n|\left|\frac{b_0}{b_N}\right|\gamma^{n-N}$$

$$\left|a_n \cdot \frac{b_1}{b_n}\right| \leq |a_n|\left|\frac{b_1}{b_N}\right|\gamma^{n-N}$$

$$= |a_{n-1}|\left|\frac{b_1}{b_N}\right|\frac{\gamma}{\gamma^N}\gamma^{n-1}$$

$k \in \{0, \dots, N-1\}$,

$$\left|\frac{a_{n-k}b_k}{b_n}\right| \leq |a_{n-k}|\frac{|b_k|}{|b_N|}\times\frac{|b_N|}{|b_n|}$$

$$\leq |a_{n-k}|\frac{|b_k|}{|b_N|}\underbrace{\gamma^{n-N}}_{\gamma^{n-k}\frac{\gamma^k}{\gamma^N}}$$

$$\frac{|a_{n-k}b_k|}{|b_n|} \leq \left|\frac{b_k}{b_N}\right| \times \frac{\gamma^k}{\gamma^N} \times |a_{n-k}|\gamma^{n-k}$$

On pose $M = \sup\limits_{0 \leq k \leq N-1} \left|\frac{b_k}{b_N}\right|\frac{\gamma^k}{\gamma^N} + 1$

$\forall k \in [\![0, N-1]\!]$,

$$\left|a_{n-k}\frac{b_k}{b_n}\right| \leq M \cdot |a_{n-k}|\gamma^{n-k}$$

$\forall p$, $\left|a_{n-p}\frac{b_p}{b_n}\right| \leq M\,|a_{n-p}|\gamma^{n-p}$

$\forall l$, $\left|a_l\frac{b_{n-l}}{b_n}\right| \leq M\,|a_l|\gamma^l$

$\left|\frac{b_N}{b_n}\right| \leq \gamma^{n-N}$

<u>Bilan</u> $\left|a_p \cdot \frac{b_{n-p}}{b_n}\right| \underset{n-p \geq N}{\leq} |a_p|\gamma^p \leq M\,|a_p|\gamma^p$

$n-p < N$, $k = n-p$, $\left|a_{n-k}\frac{b_k}{b_n}\right| \leq M\,|a_{n-k}|\,\gamma^{n-k}$

2.4 : $\sum_{n \geq 1} (l_n n)\, x^n$ | $l_n n = 1 + \frac{1}{2} + \dots + \frac{1}{n} + \underbrace{\alpha_n}_{\text{bornée}}$

$$f(x) = \underbrace{\sum_{n=1}^{+\infty} \left(1 + \dots + \frac{1}{n}\right) x^n}_{\frac{-\ln(1-x)}{1-x}} + \underbrace{\sum_{n=1}^{+\infty} \alpha_n x^n}_{|\cdot| \leq \frac{M}{1-x}}$$

$$f(x) \underset{1^-}{\sim} \frac{-\ln(1-x)}{1-x}$$

3.1 $$\frac{e^x - e^{-x}}{2} \times \frac{e^{ix} - e^{-ix}}{2i}$$

$$= \frac{e^{(1+i)x} + e^{-(1+i)x} - \left(e^{(-1+i)x} + e^{(1-i)x}\right)}{4i}$$

$$\left(\operatorname{arctg}\left(tg(a) \frac{1+x}{1-x}\right)\right)' = \frac{2}{(1-x)^2} \frac{tg(a)}{1 + tg^2(a) \left(\frac{1+x}{1-x}\right)^2}$$

$$= \frac{2\, tg(a)}{(1-x)^2 + tg^2(a)(1+x)^2} = \frac{2 \sin a \cos a}{\cos^2 a\, (1-x)^2 + \sin^2(a)(1+x)^2}$$

$$= \frac{2 \sin a \cos a}{x^2 - 2\cos 2a\, x + 1}$$

$$= \frac{\sin 2a}{(x - e^{i2a})(x - e^{-i2a})} = \frac{A}{e^{2ia} - x} + \frac{B}{e^{-2ia} - x}$$

$$|x| < 1, \quad \frac{1}{e^{2ia} - x} = \frac{1}{e^{2ia}} \times \frac{1}{1 - x e^{-2ia}} = e^{-2ia} \sum_{n=0}^{+\infty} e^{-2ian} x^n$$

$$\frac{1}{e^{-2ia} - x} = e^{2ia} \times \frac{1}{1 - x e^{2ia}} = e^{2ia} \sum_{n=0}^{+\infty} e^{2ian} x^n$$

$$-A = \frac{\sin 2a}{e^{2ia} - e^{-2ia}} = \frac{\sin 2a}{2i \sin 2a} \Rightarrow A = \frac{i}{2}$$

$-2i \sin(2a(n+1))$

par conjugaison, $B = -\frac{i}{2}$

$$f'(x) \underset{|x|<1}{=} \frac{i}{2} \sum_{n=0}^{+\infty} \left(e^{-2i(n+1)a} - e^{2ia(n+1)}\right) x^n = \sum_{n=0}^{+\infty} \sin 2(n+1)a$$

$$f(x) = a + \sum_{n=0}^{+\infty} \sin 2(n+1)a \, \frac{x^{n+1}}{n+1}$$

3.2 $I_\mu = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\mu^2t^2)}}$ $\mu \in]0,1[$

Changement de variable $t = \cos\theta$ $dt = -\sin\theta\, d\theta$

$$I_\mu = \int_0^{\frac{\pi}{2}} \frac{\sin\theta\, d\theta}{\sin\theta\sqrt{1-\mu^2\cos^2\theta}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\mu^2\cos^2\theta}}$$

$$= \int_0^{\frac{\pi}{2}} \sum_{k=0}^{+\infty} \frac{1}{4^k}\binom{2k}{k} \mu^{2k}\cos^{2k}\theta\, d\theta$$

$$\underset{(CVN)}{=} \sum_{k=0}^{+\infty} \frac{1}{4^k}\binom{2k}{k} \mu^{2k} W_{2k}$$ intégrale de Wallis

$$= \frac{\pi}{2}\sum_{k=0}^{+\infty} \frac{1}{16^k}\binom{2k}{k}^2 \mu^{2k}$$

$$I_\mu = \frac{\pi}{2}\sum_{k=0}^{+\infty} \frac{1}{16^k}\binom{2k}{k}^2 \mu^{2k}$$

Stirling : $\frac{1}{16^k}\binom{2k}{k}^2 \sim \frac{1}{\pi}\times\frac{1}{k}$

$$I_\mu \underset{1^-}{\sim} \frac{\pi}{2}\sum_{k=1}^{+\infty} \frac{1}{\pi}\times\frac{1}{k}\mu^{2k}$$

$$\underset{1^-}{\sim} -\frac{1}{2}\ln(1-\mu^2) \sim \frac{1}{2}\left(\ln\left(\frac{1}{1-\mu}\right)+\ln\left(\frac{1}{1+\mu}\right)\right)$$

$$\sim \frac{1}{2}\ln\left(\frac{1}{1-\mu}\right)$$

3.3

$$F(X) = P(X) + \sum_{(a,\alpha)\in\hat{\pi}} \frac{\lambda_{a,\alpha}}{(a-z)^\alpha}$$

$$\frac{1}{(a-z)^\alpha} = \frac{1}{a^\alpha}\left(1-\frac{z}{a}\right)^{-\alpha} \underset{\text{S.E.}}{=} \sum_{n=0}^{+\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!}\frac{z^n}{a^{n+\alpha}}$$

$|z|<|a|$ Cauchy

CL : PSE de $F(z)$ où $|z| < \min\limits_{a \text{ pôle de } F} |a|$

$$a_n = \frac{F^{(n)}(0)}{n!} \in \mathbb{K}$$

3.4

$$e^1 = \sum_{n=0}^{+\infty} \frac{1}{(3n)!} + \sum_{n=0}^{+\infty} \frac{1}{(3n+1)!} + \sum_{n=0}^{+\infty} \frac{1}{(3n+2)!}$$

$$e^{j} = \sum_{n=0}^{+\infty} \frac{1}{(3n)!} + \sum_{n=0}^{+\infty} \frac{j}{(3n+1)!} + \sum_{n=0}^{+\infty} \frac{j^2}{(3n+2)!}$$

$$e^{j^2} = \sum_{n=0}^{+\infty} \frac{1}{(3n)!} + \sum_{n=0}^{+\infty} \frac{j^2}{(3n+1)!} + \sum_{n=0}^{+\infty} \frac{j}{(3n+2)!}$$

$$e + e^{j} + e^{j^2} = 3 \times \sum_{n=0}^{+\infty} \frac{1}{(3n)!}$$

$$\sum_{n=0}^{+\infty} \frac{1}{(3n)!} = \frac{1}{3}\left(e + e^{-\frac{1}{2}}\left(e^{\frac{\sqrt{3}}{2}i} + e^{-\frac{\sqrt{3}}{2}i}\right)\right)$$

$$= \frac{1}{3}\left(e + e^{-\frac{1}{2}} \times 2\cos\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$\sum_{n\geq 1} \frac{x^n}{n(n+1)} = \sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+1}\right) x^n$$

3.5

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n S_n x^n = \left(\sum_{n=0}^{+\infty} (-1)^n x^n\right) \overbrace{\left(\sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{2n+1}\right)}^{\frac{\arctan(\sqrt{x})}{\sqrt{x}}}$$

(Les séries entières sont de rayon 1, d'où la CVA)

$$f(x) = \frac{\arctan(\sqrt{x})}{\sqrt{x}(1+x)}$$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty} (-1)^n \frac{S_n x^{n+1}}{n+1}$$

$$= \int_0^x \frac{\arctan(\sqrt{t})}{\sqrt{t}(1+t)}\,dt \overset{u=\sqrt{t}}{=} \int_0^{\sqrt{x}} \frac{2\arctan u}{u(1+u^2)} u\,du$$

$$S_n = \sum_{k=0}^{n} \frac{1}{2k+1} \qquad = \arctan^2(\sqrt{x}) \xrightarrow[x\to 1^-]{} \frac{\pi^2}{16}$$

i) $$U_n = \frac{(-1)^n}{n+1}\left(H_{2n+2} - \frac{1}{2} H_{n+1}\right)$$

$$= \frac{(-1)^n}{n+1}\left(\log(2n+2) + \gamma + O\left(\frac{1}{n}\right) - \frac{1}{2}\left(\log(n+1) + \gamma + O\left(\frac{1}{n}\right)\right)\right)$$

$\to$ série CV $\quad \frac{(-1)^n \log(n+1)}{n+1}$ CV $\quad \frac{(-1)^n \gamma}{n+1}$ CV

$O\left(\frac{1}{n(n+1)}\right)$ CV

$$-\frac{1}{n+2}S_n$$

(ii) $$|u_n|-|u_{n+1}| = \frac{1}{n+1}S_n - \frac{1}{n+2}S_{n+1} = \frac{1}{n+1}S_n + \frac{1}{n+2}(S_n - S_{n+1})$$

$$= \frac{1}{(n+1)(n+2)}S_n - \frac{1}{(2n+3)(n+2)} > 0 \text{ a.p.c.r.}$$

3.4 $$\sum_{n=0}^{+\infty} \frac{n^3-2n}{n!} z^n = \sum_{n=0}^{+\infty} \frac{n^2-2}{(n-1)!} z^n$$

$$= \sum_{n=0}^{+\infty} \frac{z^n}{(n-3)!} + 3\sum_{n=0}^{+\infty} \frac{z^n}{(n-2)!} - \sum_{n=0}^{+\infty} \frac{z^n}{(n-1)!}$$

$$= (z^3+3z^2-z)e^z$$

$$\sum_{n=1}^{+\infty} \frac{x^n}{n^2+n} = \sum_{n=1}^{+\infty} \frac{x^n}{n(n+1)} = \sum_{n=1}^{+\infty} \frac{x^n}{n} - \sum_{n=1}^{+\infty} \frac{x^n}{n+1}$$

$$= \ln(1-x)\left(1-\frac{1}{x}\right) + 1$$

3.6 $|z|<R$ $\sum_{n=0}^{+\infty} a_{pn+k} z^{np+k}$?

Idée = racines de l'unité

$\zeta = e^{\frac{2i\pi}{p}}$ $f(z)+f(\zeta z)+\dots\dots+f(\zeta^{p-1}z)$

$$= \sum_{n=0}^{+\infty} (1^n+\zeta^n+\dots+\zeta^{(p-1)n}) a_n z^n = p\sum_{n=0}^{+\infty} a_{pn} z^{pn}$$

* Pour $|z|<R$, et $k\in[\![0,p-1]\!]$, notons

$$g_k(z) = \sum_{n=0}^{+\infty} a_{pn+k} z^{pn+k} \quad (\text{rayon} \geq R)$$

On remarque que $$f(\zeta^l z) = \sum_{n=0}^{+\infty} a_n \zeta^{ln} z^n$$

$$= \sum_{k=0}^{p-1} g_k(\zeta^l z) \qquad = \sum_{k=0}^{p-1} \zeta^{lk} g_k(z)$$

Donc, $A = (\zeta^{ij})_{0\leq i,j\leq p-1}$, on a :

$$A\begin{pmatrix} g_0(z) \\ \vdots \\ \vdots \\ g_{p-1}(z) \end{pmatrix} = \begin{pmatrix} f(z) \\ f(\zeta z) \\ \vdots \\ f(\zeta^{p-1}z) \end{pmatrix} \quad \text{Or } A^{-1} = \frac{1}{p}(\zeta^{-ij})$$

donc si $k\in[\![0,p-1]\!]$, $$g_k(z) = \frac{1}{p}\sum_{j=0}^{p-1} \zeta^{-jk} f(\zeta^j z)$$

(Refaits)

2.3

a) $\forall k, \quad a_{n,k} \xrightarrow[n\to+\infty]{} l_k$

$$|a_{n,k}| \leq \alpha_k$$

HYP: $\sum \alpha_k$ converge.

Soit $n \in \mathbb{N}$.

$|a_{n,k}| \leq \alpha_k$, donc avec $\sum \alpha_k$ CV,

S_n ACV. De même, $|l_k| \leq \alpha_k$, $\sum l_k$ ACV.

On étudie $\left|S_n - \sum_{k=0}^{+\infty} l_k\right|$

$$= \left|\sum_{k=0}^{+\infty}(a_{n,k} - l_k)\right|$$

Soit $\varepsilon > 0$. Il existe $K \in \mathbb{N}$ t.q.

$$\sum_{k \geq K} \alpha_k \leq \varepsilon$$

pour $k < K$, il existe n_k t.q.

$$\forall n \geq n_k, \quad |a_{n,k} - l_k| \leq \frac{\varepsilon}{2^k}$$

Soit $N = \max(n_0, \dots, n_{K-1})$

$\forall n \geq N$, $\left|S_n - \sum_{k=0}^{+\infty} l_k\right|$

$$\leq \sum_{k=0}^{K-1} |a_{n,k} - l_k| + \sum_{k=K}^{+\infty} |a_{n,k} - l_k|$$

$$\leq \sum_{k=0}^{K-1} \frac{\varepsilon}{2^k} + \sum_{k=K}^{+\infty} 2\alpha_k$$

$$\leq 2\varepsilon + 2\varepsilon \leq 4\varepsilon.$$

Donc, $S_n \to \sum_{k=0}^{+\infty} l_k$

b) $\frac{c_n}{b_n} = \sum_{k=0}^{n} \frac{a_k b_{n-k}}{b_n}$

Soit $u_{n,k} = \frac{a_k b_{n-k}}{b_n}$ si $k \leq n$

$= 0$ sinon

Alors, $\frac{c_n}{b_n} = S_n$.

Soit $k \in \mathbb{N}$,

$$u_{n,k} \underset{n \geq k}{=} \frac{a_k b_{n-k}}{b_n} \xrightarrow[n \to +\infty]{} a_k \frac{b_{n-k}}{b_n}$$

$$\xrightarrow[n \to +\infty]{} a_k \beta^k$$

On veut $|u_{n,k}| \leq \alpha_k$, et $\sum \alpha_k < +\infty$

Soit γ t.q. $|\beta| < \gamma < 1$

N t.q. $\forall k \geq N$ $\left|\frac{b_{k-1}}{b_k}\right| \leq \gamma$

pour $n-k \geq N$, $|u_{n,k}| \leq |a_k| \gamma^k$

pour $n-k < N$, $u_{n,k} = \frac{a_k b_{n-k}}{b_n} = \frac{a_{n-l} b_l}{b_n}$ $l = n-k$

$$|u_{n,k}| = |a_{n-l}| \frac{|b_l|}{|b_N|} \frac{|b_N|}{|b_n|} \leq |a_{n-l}| \frac{|b_l|}{|b_N|} \gamma^{n-N}$$

$$\leq |a_{n-l}| \frac{|b_l|}{|b_N|} \gamma^{l-N} \gamma^{n-l}$$

Soit $M = \max_{l \in [\![0, N-1]\!]} \frac{|b_l|}{|b_N|} \gamma^{l-N}$

$$|u_{n,k}| \leq |a_k| M \gamma^k$$

Soit $M' = \max(M, 1)$

$$|u_{n,k}| \leq \underbrace{|a_k| M' \gamma^k}_{\alpha_k}$$

$$\sum a_k = M' \sum |a_k| \gamma^k \quad \text{CV car } \gamma < 1$$

D'après a),

$$\frac{c_n}{\ell_n} \to \sum_{k=0}^{+\infty} a_k \beta^k = f(\beta).$$

2.4.

$$l_n n = 1 + \frac{1}{2} + \dots + \frac{1}{n} + \alpha_n \quad , \quad \alpha_n \text{ borné}$$

$$\sum_{n=1}^{+\infty} (l_n n) x^n \qquad \alpha_n = -\gamma + o(1)$$

$$= \underbrace{\sum_{n=1}^{+\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) x^n}_{\frac{-\log(1-x)}{1-x}} + \underbrace{\sum_{n=1}^{+\infty} \alpha_n x^n}_{= -\gamma \frac{x}{1-x} + o\left(\frac{x}{1-x}\right)}$$

$$\text{en } 1^- \quad \frac{-\log(1-x) - \gamma}{1-x} + o\left(\frac{1}{1-x}\right)$$

3.1

$$\operatorname{sh} x = \frac{e^x - e^{-x}}{2} \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\operatorname{sh} x \sin x = \frac{e^{x(1+i)} - e^{(1-i)x} - e^{(i-1)x} + e^{-(1+i)x}}{4i}$$

$$= \frac{1}{4i} \sum_{k=0}^{+\infty} \frac{1}{k!} \Big[(x(1+i))^k - ((1-i)x)^k - ((i-1)x)^k + (-(1+i)x)^k \Big]$$

$$(1+i)^k - (1-i)^k - (i-1)^k + (-(1+i))^k$$

$$= \sqrt{2}^k \left(e^{i\frac{\pi}{4}k} - e^{-i\frac{\pi}{4}k} - e^{i\frac{3}{4}\pi k} + e^{-i\frac{3}{4}\pi k} \right)$$

$$= \sqrt{2}^k \left(2i \sin\left(\frac{\pi}{4}k\right) - 2i \sin\left(\frac{3}{4}\pi k\right) \right)$$

$$\operatorname{sh} x \sin x = \sum_{k=0}^{+\infty} a_k x^k \quad \text{où } a_k = \frac{1}{k!} 2^{\frac{k}{2}-1} \left(\sin\left(\frac{k}{4}\pi\right) - \sin\left(\frac{3k}{4}\pi\right) \right)$$

3.4 $\displaystyle\sum_{n\geq 1} \frac{x^n}{n^2+n} = \sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+1}\right)x^n$

$(x\neq 0)$ $\displaystyle = \sum_{n\geq 1}\frac{x^n}{n} - \frac{1}{x}\sum_{n\geq 1}\frac{1}{n+1}x^{n+1}$

$= -\ln(1-x) - \frac{1}{x}\left(-\ln(1-x) - x\right)$

$= -\ln(1-x)\left(1-\frac{1}{x}\right) + 1$

$$\sum_{n=0}^{+\infty}\frac{n^3-2n}{n!}z^n = \sum_{n=0}^{+\infty}\frac{n(n-1)(n-2)+3n^2-4n}{n!}z^n$$

$$= \sum_{n=0}^{+\infty}\frac{1}{(n-3)!}z^n + \sum_{n=0}^{+\infty}\frac{3n(n-1)-n}{n!}z^n$$

$$= z^3\sum_{n\geq 3}\frac{1}{(n-3)!}z^{n-3} + 3z^2\sum_{n\geq 2}\frac{z^{n-2}}{(n-2)!} - z\sum_{n=1}\frac{z^{n-1}}{(n-1)!}$$

$$= (z^3+3z^2-z)e^z$$

3.5 $\displaystyle\left(\sum_{n=0}^{+\infty}(-1)^n x^n\right)\left(\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{2n+1}\right)$

$$= \sum_{n=0}^{+\infty}\left((-1)^n\sum_{k=0}^{n}\frac{1}{2k+1}\right)x^n = \sum_{n=0}^{+\infty}(-1)^n S_n x^n = f(x)$$

$$\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{2n+1} = \frac{\operatorname{arctg}(\sqrt{x})}{\sqrt{x}}$$

$$f(x) = \frac{\operatorname{arctg}(\sqrt{x})}{\sqrt{x}\,(1+x)}$$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{+\infty}(-1)^n S_n\frac{x^{n+1}}{n+1}$$

$$= \int_0^x\frac{\operatorname{arctg}\sqrt{t}}{\sqrt{t}\,(1+t)}\,dt \overset{u=\sqrt{t}}{=} \int_0^{\sqrt{x}}\frac{\operatorname{arctg} u}{u(1+u^2)}2u\,du$$

$$= 2\int_0^{\sqrt{x}}\frac{\operatorname{arctg} u}{1+u^2}\,du = \left[(\operatorname{arctg} u)^2\right]_0^{\sqrt{x}} = (\operatorname{arctg}\sqrt{x})^2$$

$$\xrightarrow[x\to 1^-]{} \frac{\pi^2}{16}$$