TD : Calcul infinitésimal

1.1

$$\left|\prod_{k=1}^{n}(1+a_k)-1\right| = \left|\sum_{\substack{J\subset[\![1,n]\!]\\ J\neq\emptyset}}\ \prod_{i\in J} a_i\right|$$

$$\leq \sum_{\substack{J\subset[\![1,n]\!]\\ J\neq\emptyset}}\ \prod_{i\in J}|a_i| = \prod_{k=1}^{n}(1+|a_k|)-1$$

$$\leq e^{\sum_{k=1}^{n}|a_k|}-1$$

1.2

a) Avec 1.1, pour $m<n$,

$$|P_n-P_m| = |P_m|\times\left|\prod_{k=m+1}^{n}(1+u_k)-1\right|$$

$$\leq |P_m|\times\left(e^{\sum_{k=m+1}^{+\infty}|u_k|}-1\right)$$

De plus, pour tout n, $|P_n-1| \leq e^{\sum_{k=1}^{n}|a_k|}-1 \leq e^{\sum}-1$

Donc, (P_n) est bornée, notons par M

$$|P_n-P_m| \leq M(e^{\varepsilon}-1) \quad \text{dès que } \sum_{k=n+1}^{+\infty}|u_k|<\varepsilon$$

Par critère de Cauchy, $P_n \to P \in \mathbb{R}$.

b) Soit $\varepsilon>0$ t.q. $0<e^{\varepsilon}-1<\frac{1}{2}$

$N\in\mathbb{N}$ t.q. $\sum_{k=N+1}^{+\infty}|u_k|<\varepsilon$

$$\forall n>N,\quad |P_n-P_N| = |P_N|\times\left|\prod_{k=N+1}^{n}(1+u_k)-1\right|$$

$$\leq \frac{1}{2}|P_N|$$

$$|P-P_N| \leq \frac{1}{2}|P_N|$$

Si $P=0$, $P_N=0$, donc $\exists k\in[\![1,N]\!],\ u_k+1=0$.

1.3

a) Soit K un compact de Ω

Pour tout n, $\|u_n(z)\|_{K,\infty} = \alpha_n$, et $\sum \alpha_N < +\infty$.

Sur K,

$$|P_n(z) - 1| \leq e^{\sum_{k=1}^{n} |u_k(z)|} - 1$$

$$\leq e^{\sum_{k=1}^{n} \alpha_k} - 1$$

$$\underbrace{|P_n - P|}_{\lim\limits_{\substack{m>n \\ m \to +\infty}} |P_n - P_m|} \leq e^{A}\left(e^{\sum_{k=m+1}^{+\infty} \alpha_k} - 1\right) \xrightarrow[n \to +\infty]{CVU} 0$$

1.4

a) $$\Gamma_n(z) = \frac{n^z \cdot n!}{z(z+1)\cdots(z+n)} \longrightarrow \Gamma(z)$$

$$\Gamma_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt$$

$$\underset{t=nu}{=} \int_0^1 (1-u)^n (nu)^{z-1} n\, du$$

$$= n^z \int_0^1 (1-u)^n u^{z-1}\, du$$

$$\underset{IPP}{=} n^z \left(\underbrace{\left[(1-u)^n \frac{u^z}{z}\right]_0^1}_{=0} + n \int_0^1 (1-u)^{n-1} \frac{u^z}{z}\, du \right)$$

$$= n^z \left(\frac{n}{z}\right) \int_0^1 (1-u)^{n-1} u^z\, du$$

$$= \frac{n^z \cdot n!}{z(z+1)\cdots(z+n)}$$

b) $$\frac{1}{\Gamma_n(z)} = z n^{-z} \frac{(z+1)\dots(z+n)}{n!}$$

$$= z e^{-z \ln n} \prod_{k=1}^{n} (1 + z/k)$$

$\tilde{H}_n = 1 + \dots + \frac{1}{n} - \log n$

$$= \underbrace{z e^{z \tilde{H}_n}}_{} \prod_{k=1}^{n} (1 + z/k) e^{-\frac{z}{k}}$$

CVU sur $\bar{D}(0,R)$ par UC de e^z

$$1 + u_k(z) = (1 + z/k) e^{-\frac{z}{k}}$$

$$= (1 + \frac{z}{k}) \left(1 - \frac{z}{k} + \frac{z^2}{k^2} \underbrace{\sum_{n=2}^{+\infty} \frac{(-1)^n z^{n-2}}{k^{n-2} n!}}_{g_k(z)}\right) \quad \begin{cases} z \in \bar{D}(0,R) \\ R \geq 1 \end{cases}$$

$$|g_k(z)| \leq e^R$$

$$= 1 - \frac{z^2}{k^2} + \frac{z^2}{k^2} g_k(z) + \frac{z^3}{k^3} g_k(z)$$

$$\left| \frac{z^2}{k^2} (-1 + g_k(z)) \right| \leq \frac{R^2}{k^2} (1 + e^R)$$

Produit NCV (CC), $P_n(z) = \frac{1}{\Gamma_n(z)}$

$P_n(z) \Gamma_n(z) = 1$

$P_n(z) \to P(z) \neq 0$ $\qquad \Gamma(z) \neq 0$

$P_n(z) \to \Gamma(z)$

2.1 $$\prod_{k=1}^{n} (1 + u_k) = \sum_{I \subset [\![1,n]\!]} \prod_{i \in I} u_i$$

$$= \sum_{\underbrace{I \in \mathcal{P}([\![1,n]\!])}_{\text{suite exhaustive de } \mathcal{P}_F(\mathbb{N})}} \prod_{i \in I} u_i$$

$$\to \prod_{k=1}^{+\infty} (1 + u_k)$$

$A_n \subset \mathbb{D}$ $\quad (a_d)_{d \in D}$ sommable

A_0, $A_{n+1} \setminus A_n$ partition de $\mathbb{D}$

$$\sum_{k=1}^{n}\Big(\sum_{d\in A_{k+1}\setminus A_k} a_d\Big) + \sum_{d\in A_1} a_d \longrightarrow \sum_{\alpha\in D} a_\alpha$$

$$\sum_{d\in A_n} a_d$$

$$A_n = \{p_1^{\alpha_1}\cdots p_n^{\alpha_n}\} \quad \underbrace{\alpha_i \geq 0}_{\text{suite exhaustive}}$$

$Re(z) > 1$ $\quad \sum \frac{1}{n^z}$ est sommable

$$\sum_{k=p_1^{\alpha_1}\cdots p_n^{\alpha_n}\in A_n} p_1^{-\alpha_1 z}\cdots p_n^{-\alpha_n z} \xrightarrow[n\to+\infty]{} \zeta(z)$$

Produit de Cauchy,

$$\Big(\sum_{\alpha_i=0}^{+\infty} p_i^{-\alpha_i z}\Big)\times\cdots\times\Big(\prod_{\alpha_n=0}^{+\infty} p_n^{-\alpha_n z}\Big)$$

$$= \prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i^z}}$$

Principe de Weierstrass

$\forall k,\quad u_{n,k} \to v_k$

$\forall n,k,\quad |u_{n,k}| \leq \alpha_k$ avec $\sum \alpha_k < +\infty$

alors, $\quad S_n = \sum_{k=0}^{+\infty} u_{n,k} \longrightarrow \sum_{k=0}^{+\infty} v_k$

3.1

$$\Big(1+\frac{z}{n}\Big)^n = \sum_{k=0}^{n} C_n^k \frac{z^k}{n^k}$$

$$= \sum_{k=0}^{n} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{k \text{ fixé } \in[0,1]} \frac{z^k}{k!}$$

$n\to+\infty$ tend vers 1

$u_{n,k}(z) = 0$ si $k > n$

$u_{n,k}(z) = \Big(1-\frac{1}{n}\Big)\cdots\Big(1-\frac{k-1}{n}\Big)\frac{z^k}{k!}$ si $k \leq n$

$$\forall z \in \overline{D}(0,R) \quad \forall (n,k) \in \mathbb{N}^2 \quad \begin{cases} |u_{n,k}(z)| \leq \dfrac{R^k}{k!} = \alpha_k \\ u_{n,k}(z) \xrightarrow[n\to+\infty]{} \dfrac{z^k}{k!} \end{cases}$$

$$\left(1+\frac{z}{n}\right)^n \xrightarrow[\text{CVU sur } \overline{D}(0,R)]{} e^z$$

3.2

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} = \lim_{m\to+\infty} \underbrace{\frac{\left(1+\frac{ix}{m}\right)^m - \left(1-\frac{ix}{m}\right)^m}{2i}}_{Q_m(x)}$$

3.3,4 Si $m = 2p$.

On va mq $Q_{2p} = x \prod_{k=1}^{p-1} \left(1 - \dfrac{x^2}{4p^2 \tan^2 \frac{k\pi}{2p}}\right)$

On a $Q(0) = 0$

$Q_{2p}\left(2p \tan \frac{k\pi}{2p}\right) = 0 \quad k \in [\![-(p-1), (p-1)]\!]$

donc Q_{2p} a $2p-1$ racines distinctes.

le terme de degré 1 est $\dfrac{1}{2i}\left(2p \cdot \dfrac{ix}{2p} - 2p \dfrac{-ix}{2p}\right) = x$

D'où le résultat.

$$\left(\frac{1+z}{1-z}\right)^{2p} = 1$$

$$\Leftrightarrow z = \frac{e^{\frac{2ik\pi}{2p}} - 1}{e^{\frac{2ik\pi}{2p}} + 1} = i \operatorname{tg} \frac{k\pi}{2p}$$

4.1 $x \in]0, \pi[$,

$$\frac{Q'_{2p}(x)}{Q_{2p}(x)} = \frac{1}{x} + \sum_{k=1}^{p-1} \frac{2x}{x^2 - 4p^2 \operatorname{tg}^2 \frac{k\pi}{2p}}$$

$$\underbrace{\left| \sum_{k=1}^{p-1} \frac{1}{x^2 - 4p^2 \operatorname{tg}^2 \frac{k\pi}{2p}} - \sum_{k=1}^{+\infty} \frac{1}{x^2 - k^2\pi^2} \right|}_{\delta}$$

$\underline{N \gg 1}$, fix ε $\displaystyle\sum_{k=N}^{+\infty} \frac{1}{k^2\pi^2 - \pi^2} < \varepsilon$

$$\underline{p \gg N}, \quad |\delta| \leq \overbrace{\left| \sum_{k=1}^{N} \frac{1}{x^2 - 4p^2 \operatorname{tg}^2 \frac{k\pi}{2p}} - \sum_{k=1}^{N} \frac{1}{x^2 - k^2\pi^2} \right|}^{S_N}$$

$$+ \underbrace{\sum_{k=N+1}^{p-1} \left| \frac{1}{x^2 - 4p^2 \operatorname{tg}^2 \frac{k\pi}{2p}} \right|}_{\leq \sum_{k=N+1}^{+\infty} \frac{1}{k^2\pi^2 - \pi^2}} + \underbrace{\sum_{k=N+1}^{+\infty} \frac{1}{|x^2 - k^2\pi^2|}}_{\leq \sum_{k=N+1}^{+\infty} \frac{1}{k^2\pi^2 - \pi^2}}$$

S_N CVU sur $[0,a]$. $\to 0$

$$\frac{1}{x} + \sum_{k=1}^{p-1} \frac{2x}{x^2 - 4p^2 \operatorname{tg}^2 \frac{k\pi}{2p}} \xrightarrow[\text{compact de }]0,\pi[]{\text{CVU} /} \frac{1}{x} + \sum_{k=1}^{+\infty} \frac{2x}{x^2 - k^2\pi^2}$$

$$\underbrace{\log Q_{2p}}_{g_p} \xrightarrow[\text{CVU / compact de }]0,\pi[]{} \overbrace{\log \sin x}^{f}$$

Si $[a,b] \subset]0,\pi[$, g_p U bornée, f bornée $\Big\} \to$ à valeurs dans $[-R,R]$

(C.C.) $\forall x \in]0,\pi[$,

$$\cot g\, x - \frac{1}{x} = \sum_{k=1}^{+\infty} \frac{2x}{x^2 - k^2\pi^2}$$

$$|e^{g_p} - e^{f}| \underset{\text{A.F.}}{\leq} e^{R} \|g_p - f\|_\infty$$

$$\cot g\, x - \frac{1}{x} = \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)}{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)} - \frac{1}{x}$$

$$= \frac{1}{x} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots\right)}{\left(1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots\right)} - \frac{1}{x}$$

$$= \sum_{n=0}^{+\infty} r_{2n} x^{2n+1} \quad |x| \text{ petit}$$

$$\sum_{k=1}^{+\infty} \frac{2x}{x^2 - k^2\pi^2} = (-2x) \sum_{k=1}^{+\infty} \frac{1}{k^2\pi^2 \left(1 - \frac{x^2}{k^2\pi^2}\right)}$$

$$= (-2x) \underbrace{\left(\sum_{k=1}^{+\infty} \left(\sum_{n=0}^{+\infty} \frac{x^{2n}}{k^{2(n+1)} \pi^{2(n+1)}} \right) \right)}_{\text{famille sommable } \geq 0, \ x > 0} \quad \text{CV}$$

$$= \sum_{n=0}^{+\infty} \left(-2x^{2n+1}\right) \frac{1}{\pi^{2(n+1)}} \underbrace{\sum_{k=1}^{+\infty} \frac{1}{k^{2(n+1)}}}_{\zeta(2(n+1))}$$

On identifie : $\forall m \geq 1 \quad \frac{\zeta(2m)}{\pi^{2m}} \in \mathbb{Q}$

(Refaits)

1.4 a) $$\Gamma_n(z) = \frac{n! \cdot n^z}{z(z+1)\dots(z+n)}$$

En effet, $\Gamma_n(z) = \int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt$

$$\underset{t=nu}{=} \int_0^1 (1-u)^n (nu)^z \frac{du}{u}$$

$$= n^z \int_0^1 (1-u)^n u^{z-1}\,du$$

$$= n^z \left(\left[(1-u)^n \frac{u^z}{z}\right]_0^1 + n\int_0^1 (1-u)^{n-1} \frac{u^z}{z}\,du \right)$$

$$= n^z \cdot \frac{n}{z} \int_0^1 (1-u)^{n-1} u^z\,du$$

$$= n^z \cdot \frac{n}{z} \cdot \frac{n-1}{z+1} \int_0^1 (1-u)^{n-2} u^{z+1}\,du$$

$$= n^z \cdot \frac{n}{z} \cdot \frac{n-1}{z+1} \cdot \dots \cdot \frac{1}{z+(n-1)} \int_0^1 u^{z+(n-1)}\,du$$

$$= \frac{n!\ n^z}{z(z+1)\dots(z+n)}$$

Soit $f_n : t \longmapsto \begin{cases} \left(1-\frac{t}{n}\right)^n & \text{pour } t \in [0,n] \\ 0 & \text{pour } t > n \end{cases}$

f_n CVS vers $f : x \longmapsto e^{-x}$

Par CVD, $\Gamma_n(z) \longrightarrow \int_0^{+\infty} e^{-t} t^{z-1}\,dt = \Gamma(z)$

Donc, $\frac{1}{\Gamma(z)}$ est limite de la suite $\left(\frac{z(z+1)\dots(z+n)}{n!\ n^z}\right)$

3.1 $$\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{n^k}$$
$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{z^k}{n^k}$$
$$= \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{z^k}{k!}$$

$U_{n,k}(z) = 0$ si $k > n$
$U_{n,k}(z) = \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{z^k}{k!}$ pour $k \leq n$

pour k fixé, $U_{n,k}(z) \longrightarrow \frac{z^k}{k!}$

Sur $\overline{D}(0,R)$, $|U_{n,k}(z)| \leq \frac{R^k}{k!} = \underbrace{\alpha_k}_{\text{t.g. d'une série CV}}$

Alors, $S_n = \sum_{k=0}^{+\infty} U_{n,k}(z) \xrightarrow[\overline{D}(0,R)]{CVU} e^z$

3.2 $$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$e^{ix} = \lim_{m\to+\infty}\left(1+\frac{ix}{m}\right)^m \qquad e^{-ix} = \lim_{m\to+\infty}\left(1-\frac{ix}{m}\right)^m$$

Donc, $\sin x = \lim_{m\to+\infty} \frac{1}{2i}\left(\left(1+\frac{ix}{m}\right)^m - \left(1-\frac{ix}{m}\right)^m\right)$

3.3 pour $m = 2p$, $\left(1+\frac{iz}{2p}\right)^{2p} = \left(1-\frac{iz}{2p}\right)^{2p}$

$$\frac{1+\frac{iz}{2p}}{1-\frac{iz}{2p}} = e^{i\frac{2\pi}{2p}k} \qquad k \in [\![0, 2p-1]\!]$$

$$1+\frac{i}{2p}z = e^{i\frac{2\pi}{2p}k} - \frac{i}{2p}e^{i\frac{2\pi}{2p}k} z$$

$$z = \frac{e^{i\frac{2\pi}{2p}k} - 1}{\frac{i}{2p}\left(1+e^{i\frac{2\pi}{2p}k}\right)} = \frac{2p}{i}\, i\, tg\frac{\pi k}{2p}$$
$$= 2p\, tg\frac{k\pi}{2p}$$

Les zéros de Q_m : $2p \operatorname{tg} \frac{k\pi}{2p}$, $k \in [\![0, p-1]\!] \cup [\![p+1, 2p-1]\!]$

$2p-1$ racines !

3.4. $$Q_m(X) = \lambda \prod_{k=-p+1}^{p-1} \left(X - 2p \operatorname{tg} \frac{k\pi}{2p}\right)$$

$$\frac{1}{2i}\left(2p\left(\frac{iX}{2p}\right)^{2p-1} - 2p\left(\frac{-iX}{2p}\right)^{2p-1}\right) = \frac{p}{i}\left(i^{2p-2}\frac{X^{2p-1}}{(2p)^{2p-1}} + \frac{i^{2p-2}X^{2p-1}}{(2p)^{2p-1}}\right)$$

$$= 2p\left((-1)^{p-1}\frac{X^{2p-1}}{(2p)^{2p-1}}\right)$$

$$= \left(\frac{-1}{4p^2}\right)^{p-1} X^{2p-1}$$

$$\lambda = \left(\frac{-1}{4p^2}\right)^{p-1}$$

Donc, $$Q_m(X) = \left(\frac{-1}{4p^2}\right)^{p-1} \prod_{k=-p+1}^{p-1}\left(X - 2p \operatorname{tg}\frac{k\pi}{2p}\right)$$

$$= X \prod_{k=1}^{p-1}\left(X - 2p \operatorname{tg}\frac{k\pi}{2p}\right)\left(X + 2p \operatorname{tg}\frac{k\pi}{2p}\right)\left(\frac{-1}{4p^2}\right)$$

$$= X \prod_{k=1}^{p-1}\left(X^2 - 4p^2 \operatorname{tg}^2\frac{k\pi}{2p}\right)\left(\frac{-1}{4p^2}\right)$$

$$= X \prod_{k=1}^{p-1} \operatorname{tg}^2\frac{k\pi}{2p}\left(1 - \frac{X^2}{4p^2 \operatorname{tg}^2\frac{k\pi}{2p}}\right)$$

$$= \left(\prod_{k=1}^{p-1} \operatorname{tg}^2\frac{k\pi}{2p}\right) \times X \times \prod_{k=1}^{p-1}\left(1 - \frac{X^2}{4p^2 \operatorname{tg}^2\frac{k\pi}{2p}}\right)$$

On identifie le coefficient devant X

$$\frac{1}{2i}\left(m\frac{ix}{m} + ix\right) = x$$

Donc, $$Q_m(X) = X \prod_{k=1}^{p-1}\left(1 - \frac{X^2}{4p^2 \operatorname{tg}^2\frac{k\pi}{2p}}\right)$$