TD: Calcul différentiel

1.1. a)

Continuité : $\mathcal{C}^0$ en $\mathbb{R}^2 \setminus \{(0,0)\}$ du couple (x,y)

en $(0,0)$: $(x,0)$ $\quad \dfrac{\sin x^2}{|x|} \xrightarrow[x\to 0]{} 0$

(x,x) $\quad \dfrac{2\sin(x^2)}{\sqrt{2x^2}} = \sqrt{2}\dfrac{\sin(x^2)}{\sqrt{x^2}} \xrightarrow[x\to 0]{} ?$

(x,x^2) $\quad \dfrac{\sin(x^2)+\sin(x^4)}{\sqrt{x^2+x^4}}$

$$\frac{(\sin(x^2)+\sin(y^2))}{\sqrt{x^2+y^2}} = \frac{x^2+y^2+o(x^2+y^2)}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}+o\left(\sqrt{x^2+y^2}\right) \xrightarrow[\to(0,0)]{} 0$$

D'où, $\mathcal{C}^0$ du couple (x,y) sur $\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = \frac{2\cos(x^2)x\sqrt{x^2+y^2} - (\sin x^2+\sin y^2)\times\frac{1}{2}\times\frac{2x}{\sqrt{x^2+y^2}}}{(x^2+y^2)}$$

$$= \frac{x\left(2\cos(x^2)(x^2+y^2) - (\sin x^2+\sin y^2)\right)}{(x^2+y^2)^{3/2}}$$

$|f(x,y)| \leq \|(x,y)\|_2^2$

e) Sur $\mathbb{R}^2 \setminus \{(0,0)\}$, $\mathcal{C}^\infty$ par les opérations.

$\dfrac{\partial f}{\partial x}(0,0)$? $\quad \dfrac{f(x,0)-f(0,0)}{x} \underset{x\neq 0}{=} x\sin\dfrac{1}{|x|} \xrightarrow[x\to 0]{} 0$

$\exists \ \dfrac{\partial f}{\partial y}(0,0) = 0$ par symétrie

$f(x,y) = o\left(\|(x,y)\|_2\right)$ $\quad \boxed{\exists\ df_{(0,0)} = 0}$

$$\frac{\partial f}{\partial x} = (2x)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)\times\left(-\frac{1}{2}\right)\times\frac{2x}{(x^2+y^2)^{3/2}}$$

$$= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$ $\quad \boxed{\text{non } \mathcal{C}^1}$

$\begin{cases} f(x,0) = x^2\sin\frac{1}{x}, \ 0 \text{ en } 0 \\ x \geq 0 \end{cases}$

a) $|f(x,y)| \leq \|(x,y)\|$: $\mathcal{C}^0$

DP en $(0,0)$ $\frac{f(x,0)}{x} = \frac{\sin x^2}{x|x|} \nearrow 1 \quad \rightarrow -1$

pas de DP en $(0,0)$

1.2 On note la fonction $g : x \mapsto f(x) - f(a) - L(x-a)$

Sur $U \setminus \{a\}$, $\exists\, dg_x = df_x - L$

Soit $\varepsilon > 0$.

Soit $r > 0$, sur $\overline{B}(a,r)$, $|||dg_x||| \leq \varepsilon$

$\forall x \in \overline{B}(a,r)$

Soit $y \in]a, x]$

$\|g(x) - g(y)\| \leq \varepsilon \|x - y\|$

HYP: df est $\mathcal{C}^0$ sur $U \setminus \{a\}$

Comme g est $\mathcal{C}^0$, on passe à la limite

$y \to a$ $\quad \|g(x) - g(a)\| \leq \varepsilon \|x - a\|$

$f(x) - f(a) - L(x-a) = o(x-a)$

$\exists\ df_a = L$

Exo-supplémentaire :

$f : \Omega \xrightarrow{\mathcal{C}^\infty} \mathbb{R}$

$i < j$

$g_{i,j}(X) = \frac{f(x_1, \dots, x_i, x_j, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_i, \dots, x_n)}{x_i - x_j}$

si $x_i \neq x_j$

si $x_i = x_j$ $\quad g_{i,j}(X) = \frac{\partial f}{\partial x_j}(X)$

MQ : g est de classe $\mathcal{C}^\infty$.

$g_{i,j}(X) = \int_0^1 \frac{\partial f}{\partial x_j}(x_1, \dots, x_i, \dots, x_i + t(x_j - x_i), \dots, x_n)\, dt$

$t \xrightarrow{\varphi} f(x_1, \dots, x_i, \dots, x_i + t(x_j - x_i), \dots, x_n)$

$\varphi'(t) = (x_j - x_i) \frac{\partial f}{\partial x_j}(x_1, \dots, x_n)$

$\int_0^1 \frac{\partial f}{\partial x_j}(\dots)\, dt = \int_0^1 \frac{\varphi'(t)}{x_j - x_i}\, dt = \frac{1}{x_j - x_i}\big(f(x_1, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_i, \dots, x_n)\big)$

$= g_{i,j}(X)$

IP : $g_{i,j}$ possède des DP
C^0 des IP : $g_{i,j}$ est C^∞.

1.3 $f(x) - f(y) = O(\|x-y\|^2) = o(\|x-y\|)$
donc $df_x = 0$ pour tout x car $\mathbb{R}^p$ CPA
donc $df = 0$
Alors, f est constante ; car f est continue.

$x_n \longrightarrow a \in \mathbb{R}^p$, $\|f(x_n) - f(x_m)\| < \varepsilon^2 \ \forall n, m \geq n_0$
$f(x_n)$ converge car de Cauchy
$f(a) = \lim_{x_n \to a} f(x_n)$, f continue.
$\boxed{\text{donc } f \text{ constante}}$, .

$$\sum n \left(\frac{\|x-y\|}{n}\right)^2 = \frac{\|x-y\|^2}{n} \xrightarrow[n\to+\infty]{} 0$$

1.4 (i) Par hypothèse, f est différentiable en x.
$\forall h \in \mathbb{R}^n$,
$$\|x_0 + h - a\|_2 - \|x_0 - a\|_2 = \langle \nabla f(x_0) | h \rangle + o(h)$$

$h = t\varepsilon_i$
$$\langle \nabla f(x_0) | t\varepsilon_i \rangle \leq \|x_0 + t\varepsilon_i - a\| - \|x_0 - a\| + o(t\varepsilon_i)$$

$$\langle \nabla f(x_0) | \varepsilon_i \rangle \leq \frac{\sqrt{\|x_0 - a\|^2 + 2t\langle \varepsilon_i | x_0 - a\rangle + t^2} - \|x_0 - a\|_2}{t} + \varepsilon(t\varepsilon_i)$$

$$\leq \frac{\langle \varepsilon_i | x_0 - a\rangle}{\|x_0 - a\|_2}$$

$t < 0$

$$\langle \nabla f(x_0) | t\varepsilon_i \rangle = \|x_0 - t\varepsilon_i - a\|_2 - \|x_0 - a\| + o(t\varepsilon_i)$$

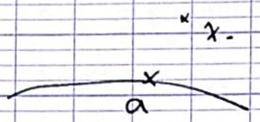
$$\langle \nabla f(x_0) | \varepsilon_i \rangle \geq \frac{\|x_0 - a\|_2 - \|x_0 - t\varepsilon_i - a\|_2}{t} + \varepsilon(t\varepsilon_i)$$

$$\frac{\langle \varepsilon_i | x_0 - a \rangle}{\|x_0 - a\|}$$

$$\forall i \in [\![1, n]\!], \quad \langle \nabla f(x_0) | \varepsilon_i \rangle = \frac{\langle x_0 - a | \varepsilon_i \rangle}{\|x_0 - a\|_2}$$

$$\Rightarrow \quad \nabla f(x_0) = \frac{x_0 - a}{\|x_0 - a\|_2} = \frac{x_0 - a}{d_A(x_0)}$$

<u>Variante</u> : X_X $\quad \lambda \in \Delta(x_0, \vec{u})$ $\qquad d_A(X) \le \|X - a\|$



Si $d(X, b) = d_A(X)$, il vient :

$$\|X_0 - b\| \le \|X_0 - X\| + \|X - b\|$$
$$\|X_0 - a\| \le \|X_0 - b\| \le \|X_0 - X\| + \|X - b\|$$
$$\underbrace{\|X_0 - a\| - \|X_0 - X\|}_{\|X_0 - a\|} \le \|X - b\|$$

d_A est diff en X,

$$d_A \text{ 1-lipsch} \Rightarrow \|\underbrace{\vec{\nabla} d_A(X_0)}_{f}\| \le 1$$

$$|\langle \vec{\nabla} f(X_0) \,|\, th\rangle + o(th)| = |f(X_0 + th) - f(X_0)|$$
$$\le \|th\|$$

$t \to 0^+$, $\quad |\langle \vec{\nabla} f(X_0) \,|\, h\rangle| \le 1$

$$f(X) - f(X_0) = -\|X - X_0\| = \langle X - X_0 \,|\, w_0\rangle$$
$$w_0 = \frac{X_0 - a}{\|X_0 - a\|}$$

$$\langle \underbrace{w}_{\|.\| \le 1} \,|\, X - X_0\rangle + o(X - X_0) = \langle X - X_0 \,|\, \underbrace{w_0}_{\|.\| = 1}\rangle$$

$$X - X_0 = \lambda w_0, \quad \cancel{\lambda}\langle \underbrace{w}_{\|.\| \le 1} \,|\, \underbrace{w_0}_{\|.\| \le 1}\rangle + o(\lambda) = \langle \cancel{\lambda} w_0 \,|\, w_0\rangle = 1$$

iii) p est 1-lipschitzienne

$$\forall Y \in A, \quad \langle X - P(X) \,|\, Y - P(X)\rangle \le 0$$
$$\langle X - P(X) \,|\, \underbrace{P(X')}_{Y} - P(X)\rangle \le 0$$
$$\Leftrightarrow \langle X' - P(X') \,|\, P(X) - P(X')\rangle \le 0$$
$$\langle X - X' + P(X') - P(X) \,|\, P(X') - P(X)\rangle \le 0$$

$$\|P(X)-P(X')\|^2 \leq \langle X-X' | P(X)-P(X')\rangle$$
$$\leq \|X-X'\| \times \|P(X)-P(X')\|$$
$$\|P(X)-P(X')\| \leq \|X-X'\|$$

1.5 On note $I = \{ i \in [\![1,k]\!], \mid f_i(a) = \varphi(a)\}$

Si $j \notin I$, $f_j(a) > \varphi(a)$, par $\mathcal{C}^0$, au $v(a)$,

$f_j(x) > \varphi(x)$

On suppose SNG, $I = [\![1,p]\!]$.

$\underline{n=1}$: on indexe, $f_1'(a) \leq \dots \leq f_p'(a)$

Si $f_1'(a) < f_p'(a)$

$\varphi(a+\lambda) = \varphi(a) + \inf(\lambda f_1'(a) + o_1(\lambda), \dots, \lambda f_p'(a) + o_p(\lambda))$

Par ex $f_p'(a) > 0$, $\lambda < 0$,

$\varphi(a+\lambda) = \varphi(a) + \lambda f_p'(a) + o(\lambda)$

$f_1'(a) > 0$, $\lambda > 0$, $\varphi(a+\lambda) - \varphi(a) = \lambda f_1'(a) + o(\lambda)$

C.N : $f_1'(a) = f_p'(a)$ $\qquad \varphi_d'(a) \neq \varphi_g'(a)$

$\underline{n \geq 2}$: Par ex : $u_i = \nabla f_i(a)$

$\|u_j\| = \max_{1 \leq i \leq k} \|u_i\|$

1) $u_1 = \dots = u_n$

$\varphi(a+h) - \varphi(a) = \inf_{1 \leq i \leq k} \{\langle u_i | h\rangle + \underbrace{o_i(h)}\} = \langle u_1 | h\rangle + o(h)$

$\pm \|h\|$ ~~déf~~ ~~tou~~

2) On suppose φ différentiable en a.

$\|u_1\|^2 \geq \langle u_1 | u_i\rangle$ $\qquad h = \lambda u_1$, $\varphi(a+h) - \varphi(a) = \inf_{1 \leq i \leq k} \{\lambda \langle u_1 | u_i\rangle + o(\lambda)\}$

$\lambda < 0$, $\varphi(a+h) - \varphi(a) = \lambda \|u_1\|^2 + o(\lambda)$

$\lambda > 0$, $\varphi(a+h) - \varphi(a) = \inf\{\lambda \|u_1\|^2 + o_1(\lambda), \dots, \lambda \langle u_1 | u_k\rangle + o_k(\lambda)\}$

$= \lambda \|u_1\|^2 + o(\lambda)$

φ différentiable, $\Rightarrow$ dérivée selon u_1

Néc (ABS), $\langle u_1 | u_i\rangle = \|u_1\|^2$ $\Big\}$ $\boxed{u_i = u_1}$

CNS : $\nabla f_1(a) = \nabla f_2(a) = \dots = \nabla f_p(a)$

$\mathcal{C}^0$

2.1 $\qquad \dfrac{|xy|}{x^2+y^2} \leq \dfrac{1}{2} \qquad |f(x,y)| \leq (x^2+y^2)$

$$= \|(x,y)\|_2^2 = o(\|(x,y)\|)$$

$$\exists\, df_{(0,0)} = 0$$

$$\frac{\partial f}{\partial x} = y\,\frac{x^2-y^2}{x^2+y^2} + xy\underbrace{\left(\overbrace{\frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2}}^{4xy^2}\right)}$$

$$|\quad| \leq |y|$$

$$\frac{\partial f}{\partial x} \xrightarrow[(0,0)]{} 0$$

$$\frac{\partial f}{\partial x} = y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{4xy^2}{(x^2+y^2)^2} = \frac{y(x^4-y^4)+4x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\left|\frac{\partial f}{\partial x}\right| \leq |y| \qquad$$ ~~continue sur~~

$$\frac{\partial f}{\partial x}_{(0,0)} = \frac{f(x,0)-f(0,0)}{x} = 0 \;,\quad \frac{\partial f}{\partial x}\ \mathcal{C}^0$$

$$\frac{\partial^2 f}{\partial x^2}_{(0,0)} = \frac{0-0}{x-0} = 0 \;,\quad \exists\ \frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

$$\frac{\partial^2 f}{\partial x^2} = y\,\frac{(4x^3+8xy^2)(x^2+y^2)^2 - (x^4+4x^2y^2-y^4)\times 2(x^2+y^2)\times 2x}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial x^2}(t,t) = t \times \frac{12t^3 \times 4t^4 - 4t^4 \times 4t \times 2t^2}{16t^8}$$

$$= \frac{48-32}{16} = 1 \qquad \frac{\partial^2 f}{\partial x^2}(t,t) = 1$$

$\dfrac{\partial^2 f}{\partial x^2}$ n'est pas $\mathcal{C}^0$ en $(0,0)$

D'où, f n'est pas $\mathcal{C}^2$.

2.2

$$f(x,y) = g(u,v) = f\left(\sqrt{uv}, \sqrt{\frac{u}{v}}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial g}{\partial u} y + \frac{\partial g}{\partial v}\frac{1}{y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial u^2} y^2 + \frac{\partial^2 g}{\partial v^2}\frac{1}{y^2} + \frac{\partial^2 g}{\partial v \partial u} + \frac{\partial^2 g}{\partial u \partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial u} x + \frac{\partial g}{\partial v}\left(-\frac{x}{y^2}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial u^2} x^2 + \frac{\partial^2 g}{\partial v^2}\left(\frac{x^2}{y^4}\right) + \frac{\partial^2 g}{\partial v \partial u}\left(-\frac{x}{y^2}\right)x + \frac{\partial^2 g}{\partial u \partial v}\left(-\frac{x^2}{y^2}\right)$$
$$+ \frac{\partial g}{\partial v}\left(\frac{2x}{y^3}\right)$$

$$0 = x^2\frac{\partial^2 f}{\partial x^2} - y^2\frac{\partial^2 f}{\partial y^2}$$

$$= \frac{\partial^2 g}{\partial u^2} u^2 + \frac{\partial^2 g}{\partial v^2} v^2 + 2\frac{\partial^2 g}{\partial u \partial v} uv$$
$$- \left(\frac{\partial^2 g}{\partial u^2} u^2 + \frac{\partial^2 g}{\partial v^2} v^2 - 2\frac{\partial^2 g}{\partial u \partial v} uv + 2\frac{\partial g}{\partial v} v\right)$$

$$= 4\frac{\partial^2 g}{\partial u \partial v} uv - 2\frac{\partial g}{\partial v} v$$

v s'annule jamais, $\quad \frac{\partial g}{\partial v} = 2\frac{\partial^2 g}{\partial u \partial v} u$

$$\cancel{\frac{\partial\left(u\frac{\partial g}{\partial v}\right)}{\partial u} = \frac{\partial g}{\partial v} + \frac{\partial^2 g}{\partial u \partial v}}$$

$$\left(\frac{\frac{\partial g}{\partial v}}{\sqrt{u}}\right)'_u = \cancel{\frac{\frac{\partial^2 g}{\partial u \partial v} u^2 - \frac{\partial g}{\partial v}}{u^4}} \qquad \frac{\frac{\partial^2 g}{\partial u \partial v}\sqrt{u} - \frac{1}{2\sqrt{u}}\frac{\partial g}{\partial v}}{u}$$

$$= \frac{2u\frac{\partial^2 g}{\partial u \partial v} - \frac{\partial g}{\partial v}}{2u^{\frac{3}{2}}} = 0$$

$$\frac{\frac{\partial g}{\partial v}}{\sqrt{u}} = h(v) \qquad \frac{\partial g}{\partial v} = h(v)\sqrt{u}$$

$$C \in \mathbb{R}$$

$$g(u,v) = \sqrt{u}\, H(v) + C, \quad H \in \mathcal{C}^2(]0,+\infty, \mathbb{R})$$

$$f(x,y) = \sqrt{xy}\, H\left(\frac{x}{y}\right) + C$$

On examine:

$$\frac{\partial f}{\partial x} = \frac{1}{2}\sqrt{\frac{y}{x}}\, H\left(\frac{x}{y}\right) + \sqrt{xy}\, H'\left(\frac{x}{y}\right)\frac{1}{y}$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4}\sqrt{\frac{y}{x^3}}\, H\left(\frac{x}{y}\right) + \frac{1}{2}\sqrt{\frac{y}{x}}\, H'\left(\frac{x}{y}\right)\frac{1}{y} + \frac{1}{2}\times\frac{1}{\sqrt{xy}} H'\left(\frac{x}{y}\right)$$
$$+ \sqrt{\frac{x}{y}}\, H''\left(\frac{x}{y}\right)\frac{1}{y}$$

$-\sqrt{\frac{x^3}{y^3}}$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}}\, H\left(\frac{x}{y}\right) + \sqrt{xy}\, H'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{4}\sqrt{\frac{x}{y^3}}\, H\left(\frac{x}{y}\right) + \frac{1}{2}\sqrt{\frac{x}{y}}\, H'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) + \frac{3}{2}\,\frac{\sqrt{x^3}}{y^{\frac{5}{2}}} H'\left(\frac{x}{y}\right)$$
$$- \sqrt{\frac{x^3}{y^3}}\, H''\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)$$

$$x^2\frac{\partial^2 f}{\partial x^2} = -\frac{1}{4}\sqrt{xy}\, H\left(\frac{x}{y}\right) + \sqrt{\frac{x^3}{y}}\, H'\left(\frac{x}{y}\right) + \sqrt{\frac{x^5}{y^3}}\, H''\left(\frac{x}{y}\right)$$

$$y^2\frac{\partial^2 f}{\partial y^2} = -\frac{1}{4}\sqrt{xy}\, H\left(\frac{x}{y}\right) + \sqrt{\frac{x^3}{y}}\, H'\left(\frac{x}{y}\right) + \sqrt{\frac{x^5}{y^3}}\, H''\left(\frac{x}{y}\right)$$

D'où, $f(x,y) = \sqrt{xy}\, H\left(\frac{x}{y}\right) + C$ avec $H \in \mathcal{C}^2(]0,+\infty[, \mathbb{R})$

$2u\frac{\partial^2 g}{\partial v \partial u} - \frac{\partial g}{\partial v}$

$$\frac{\partial}{\partial v}\left(2u\frac{\partial g}{\partial u} - g(u,v)\right) = 0$$

$$\Rightarrow 2u\frac{\partial g}{\partial u} - g(u,v) = h(u)$$

On fixe $v \in \mathbb{R}$, $g_v(u) = \lambda(v)\sqrt{u}$

$$g_v(u) = \sqrt{u}\int_1^u \frac{h(t)}{\sqrt{t}}\, dt$$

2.1 Si $\mathcal{C}^2$ $\gamma(t) = (tx, ty)$ $F(1) = f \circ \gamma(t)$

$= f(tx, ty)$

Taylor : $F(1) = f(x,y) = \underbrace{F(0)}_{=0} + \underbrace{F'(1)}_{df(0)\gamma'(0)=0} + \int_0^1 F''(t)(1-t)\,dt$

$F''(t) = F''(t) - F''(0) + F''(0)$

$\int_0^1 F''(t)(1-t)\,dt$

$F''(t) = \frac{\partial^2 f}{\partial x^2}(\gamma(t))^2 x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\gamma(t)) + \frac{\partial^2 f}{\partial y^2}(\gamma(t))\, y^2$

$xy\,\frac{x^2-y^2}{x^2+y^2} = ax^2 + 2bxy + cy^2 + o(\|(x-y)\|^2)$

$xy(x^2-y^2) = (ax^2+2bxy+cy^2)(x^2+y^2) + o(\|(x-y)\|^2(x^2+y^2))$

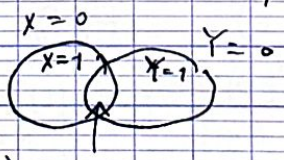
3.1 $X \sim \mathcal{B}(p)$, $Y \sim \mathcal{B}(q)$ avec $p, q \in [0,1]$

$\mathbb{P}(X=Y) = \mathbb{P}(X=Y=0) + \mathbb{P}(X=Y=1)$

$= pq + (1-p)(1-q)$ si X, Y indépendantes

$= 1 - (p+q) + 2pq$

$p \leq q$



$(p+q-1) \leq \quad \leq p$

$\underline{p=q}$: $f(p,q) = 1 - 2p + 2p^2$ min en $\frac{1}{2}$

$p+q=1$ $f(p,q) = 2pq$ max en $\frac{1}{2}$

f n'admet pas d'extremum à l'intérieur.

2.3 Soit $x \in \mathbb{R}^n$, $t \in [0,1]$, on pose
$$\varphi(t) = f(tx)$$
* φ est de classe $\mathcal{C}^2$
* La formule de Taylor reste-intégrale :
$$f(x) = \varphi(1) = \varphi(0) + \varphi'(0) + \int_0^1 (1-t)\varphi''(t)\,dt$$
$$\varphi'(t) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx), \quad \varphi'(0) = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(0) = 0$$
$$\varphi''(t) = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} x_j \frac{\partial^2 f}{\partial x_j \partial x_i}(tx)$$
(Schwarz)
$$= \sum_{1 \le i,j \le n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(tx)$$

On pose $g_{i,j}(x) = \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(tx)\,dt$

- g_{ij} est de classe $\mathcal{C}^{n-2}$
- Schwarz $\Rightarrow$ $g_{ij} = g_{ji}$
- $g_{ij}(0) = \int_0^1 (1-t)\, 2 a_{ij} = a_{ij}$

2.1 On suppose que f est $\mathcal{C}^2$. Soit $(x,y) \in \mathbb{R}^2$
On pose $\gamma(t) = (tx, ty)$
$$F(t) = f \circ \gamma(t) \qquad F(1) = f(x,y)$$
$$F(1) = F(0) + F'(0) + \int_0^1 F''(t)(1-t)\,dt$$

$$F(0) = f(0,0) = 0$$
$$F'(0) = df_{(0,0)}(x,y) = 0$$

(Refaits)

1.1 e) $f(x,y) = (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$

f est C^∞ sur $\mathbb{R}^2 \setminus \{(0,0)\}$

$\to$ $|f(x,y)| \leq x^2+y^2$

D'où C^0 en $(0,0)$.

$\to$ $f(x,y) - f(0,0) = O(\|(x,y)\|_2^2) = o(\|(x,y)\|_2)$

D'où, il existe $df_{(0,0)} = 0$.

$\to$ $\frac{\partial f}{\partial x} = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) + (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) \times \left(-\frac{2x}{2(x^2+y^2)^{\frac{3}{2}}}\right)$

$= 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$

$\frac{\partial f}{\partial x}_{(0,0)} = \frac{f(x,0)-f(0,0)}{x-0}$

$= x\sin\left(\frac{1}{|x|}\right) \xrightarrow[(0,0)]{} 0$, $\frac{\partial f}{\partial x}_{(0,0)} = 0$

Donc, non C^1

a) $f(x,y) = \frac{\sin x^2 + \sin(y^2)}{\sqrt{x^2+y^2}}$

$|f(x,y)| \leq \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2}$

$\to$ C^0 sur $\mathbb{R}^2$.

$\to$ $\frac{\partial f}{\partial x} = \frac{(\cos x^2)2x\sqrt{x^2+y^2} - (\sin x^2+\sin y^2)\times\frac{1}{2\sqrt{x^2+y^2}}\times 2x}{(x^2+y^2)}$

$= \frac{2(\cos x^2)x(x^2+y^2) - (\sin x^2+\sin y^2)x}{(x^2+y^2)^{\frac{3}{2}}}$

$\left|\frac{\partial f}{\partial x}\right| \leq \frac{2|x|+|x|}{\sqrt{x^2+y^2}}$

$\lim_{\substack{x\neq 0\\ x\to 0}} \frac{f(x,0)}{x} = \lim_{\substack{x\neq 0\\ x\to 0}} \frac{\sin x^2}{x|x|}$ $\xrightarrow[x<0]{x\to 0} -1$, $\xrightarrow[x>0]{x\to 0} 1$

pas de DP en $(0,0)$, pas de différentiel en $(0,0)$