TD: Asymptotique, Intégrales

1.3

Soit $\tan U_n = U_n$, $U_n \in]n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}[$

$n\pi$ $\quad U_n \sim n\pi.$

Posons $V_n = U_n - n\pi$

$\tan U_n = \tan V_n = U_n$

$V_n = \arctan U_n$

$n\pi + \frac{\pi}{2}$ $\quad V_n = \frac{\pi}{2} - \underbrace{\arctan \frac{1}{U_n}}_{\to 0} \to \frac{\pi}{2}$

Posons $W_n = U_n - (n\pi + \frac{\pi}{2})$

$$W_n = -\arctan \frac{1}{U_n}$$

$$= -\arctan \frac{1}{n\pi + \frac{\pi}{2} + o(1)}$$

$$= -\frac{1}{n\pi + \frac{\pi}{2} + o(1)} + \frac{1}{3}\left(\frac{1}{n\pi + \frac{\pi}{2} + o(1)}\right)^3 + o\left(\frac{1}{n^3}\right)$$

$$= -\frac{1}{n\pi}\left(1 + \frac{1}{2n} + o\left(\frac{1}{n}\right)\right)^{-1} + \frac{1}{3\pi^3 n^3} + o\left(\frac{1}{n^3}\right)$$

$n\pi + \frac{\pi}{2} - \frac{1}{n\pi}$

$$= -\frac{1}{n\pi} + \frac{1}{2n^2\pi} - \frac{1}{n\pi \times 4n^2} + o\left(\frac{1}{n^2}\right)$$

$$= -\frac{1}{\pi n} + \frac{1}{2\pi n^2} + o\left(\frac{1}{n^2}\right)$$

$$U_n = n\pi + \frac{\pi}{2} - \frac{1}{\pi n} + \frac{1}{2\pi n^2} + o\left(\frac{1}{n^2}\right)$$

2.2 en 0 :

Soit $\varepsilon > 0$. Il existe $\eta > 0$ t.q. $\forall x \in [0, \eta]$, $\cos x \geq 1 - \varepsilon$.

Soit $x \leq \frac{\eta}{3}$, $\int_x^{3x} \frac{\cos t}{t} dt \leq \int_x^{3x} \frac{1}{t} dt = [\ln t]_x^{3x} = \ln 3$

$$\int_x^{3x} \frac{\cos t}{t} dt \geq 1 - \varepsilon \int_x^{3x} \frac{1}{t} dt = \ln 3 \, (1-\varepsilon)$$

Donc $\int_x^{3x} \frac{\cos t}{t} dt \xrightarrow[x \to 0]{} \ln 3$.

en $+\infty$:

$$\int_x^{3x} \frac{\cos t}{t} dt$$

$$= \underbrace{\left[\frac{\sin t}{t}\right]_x^{3x}}_{\longrightarrow 0} + \underbrace{\int_x^{3x} \frac{\sin t}{t^2} dt}$$

$$\left| \quad \right| \leq \int_x^{3x} \frac{1}{t^2} dt = \frac{1}{x} - \frac{1}{3x} = \frac{2}{3x} \longrightarrow 0$$

Donc $\int_x^{3x} \frac{\cos t}{t} dt \xrightarrow[+\infty]{} 0$.

1.1.

a) $(1+x)^{1/x^2} = e^{\frac{1}{x^2} \ln(1+x)}$

$$= e^{\frac{1}{x^2}\left(\ln x + \ln\left(1 + \frac{1}{x}\right)\right)}$$

$$= e^{\frac{\ln x}{x^2}} \times e^{\frac{1}{x^2}\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + o\left(\frac{1}{x^3}\right)\right)}$$

$\frac{1}{x^3} - \frac{1}{2x^4} + o\left(\frac{1}{x^5}\right)$

$$= \left(1 + \frac{\ln x}{x^2} + \frac{(\ln x)^2}{2x^4} + o\left(\frac{1}{x^5}\right)\right)\left(1 + \frac{1}{x^3} - \frac{1}{2x^4} + o\left(\frac{1}{x^5}\right)\right)$$

$$= 1 + \frac{1}{x^3} - \frac{1}{2x^4} + \frac{\ln x}{x^2} + \frac{\ln x}{x^5} + \frac{(\ln x)^2}{2x^4} + o\left(\frac{1}{x^5}\right)$$

$$= 1 + \frac{\ln x}{x^2} + \frac{1}{x^3} + o\left(\frac{1}{x^3}\right)$$

$$\ln(\ln(1+x))$$
$$= \ln\left(\ln x + \ln\left(1+\frac{1}{x}\right)\right)$$
$$= \ln(\ln x) + \ln\left(1+\frac{\ln\left(1+\frac{1}{x}\right)}{\ln x}\right)$$
$$= \ln(\ln x) + \ln\left(1+\frac{\frac{1}{x}-\frac{1}{2x^2}+O\left(\frac{1}{x^3}\right)}{\ln x}\right)$$
$$= \ln(\ln x) + \ln\left(1+\frac{1}{x\ln x}-\frac{1}{2x^2\ln x}+O\left(\frac{1}{x^3\ln x}\right)\right)$$

$$\frac{1}{x^2(\ln x)^2}$$

$$= \ln(\ln x) + \frac{1}{x\ln x} - \frac{1}{2x^2\ln x} + O\left(\frac{1}{x^3\ln x}\right) + \frac{1}{2x^2(\ln x)^2} + O\left(\frac{1}{x^2(\ln x)^2}\right)$$
$$= \ln(\ln x) + \frac{1}{x\ln x} - \frac{1}{2x^2\ln x} + o\left(\frac{1}{x^2\ln x}\right)$$

$$(x+1)^{\frac{1}{x+1}} - x^{\frac{1}{x}}$$
$$= x^{\frac{1}{x+1}}\left(1+\frac{1}{x}\right)^{\frac{1}{x+1}} - x^{\frac{1}{x+1}}\, x^{\frac{1}{x(x+1)}}$$
$$= x^{\frac{1}{x+1}}\left(e^{\frac{1}{x+1}\ln\left(1+\frac{1}{x}\right)} - e^{\frac{1}{x(x+1)}\ln x}\right)$$

$$\frac{1}{x+1} = \frac{1}{x}\left(1-\frac{1}{x}+\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right)$$

$$= x^{\frac{1}{x+1}}\left(e^{\frac{1}{x+1}\left(\frac{1}{x}-\frac{1}{2x^2}+\frac{1}{3x^3}+O\left(\frac{1}{x^4}\right)\right)} - e^{\frac{\ln x}{x(x+1)}}\right)$$
$$= x^{\frac{1}{x+1}}\left(e^{\frac{1}{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right)\left(1-\frac{1}{2x}+\frac{1}{3x^2}+O\left(\frac{1}{x^3}\right)\right)} - e^{\frac{\ln x}{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right)}\right)$$

$$\frac{4}{3}-\frac{1}{2}=\frac{5}{6}$$

$$1-\frac{1}{x}+\frac{1}{x^2}$$
$$-\frac{1}{2x}-\frac{1}{2x^2}$$
$$\frac{1}{3x^2}$$

$$= x^{\frac{1}{x+1}}\left(e^{\frac{1}{x^2}\left(1-\frac{3}{2x}+\frac{5}{6x^2}+O\left(\frac{1}{x^3}\right)\right)} - e^{\frac{\ln x}{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}+O\left(\frac{1}{x^3}\right)\right)}\right)$$
$$= x^{\frac{1}{x+1}}\left(1+\frac{1}{x^2}-\frac{3}{2x^3}+\frac{5}{6x^4}+O\left(\frac{1}{x^5}\right)+\frac{1}{2x^4}-\frac{3}{2x^5}+O\left(\frac{1}{x^6}\right)\right.$$
$$\left.-\left(1+\frac{\ln x}{x^2}-\frac{\ln x}{x^3}+\frac{\ln x}{x^4}+O\left(\frac{\ln x}{x^5}\right)+\frac{(\ln x)^2}{2x^4}+O\left(\frac{(\ln x)^2}{x^5}\right)\right)\right)$$

$$\frac{1}{2}\frac{1}{x^4}\left(1-\frac{3}{x}+O\left(\frac{1}{x^2}\right)\right)$$

$$= x^{\frac{1}{x+1}}\left(\frac{1}{x^2}-\frac{3}{2x^3}+\frac{8}{6x^4}+O\left(\frac{1}{x^5}\right)-\frac{\ln x}{x^2}+\frac{\ln x}{x^3}-\frac{\ln x}{x^4}+O\left(\frac{\ln x}{x^5}\right)\right)$$

$$-\frac{\ln x}{x^2}+\frac{1}{x^2}-\frac{(\ln x)^2}{x^3}+\frac{\ln x}{x^3}-\frac{3}{2x^3}+o\left(\frac{(\ln x)^3}{x^3}\right)$$

1.2

a) Soit $P \in \mathbb{R}[X]$ de degré $d \geq 0$.

$$P_n = X^n - P$$

pour $n \geq d+2$ $\quad P_n' = nX^{n-1} - P'$

* $P_n(1) < 0$, $\quad P_n(x) \xrightarrow[x \to +\infty]{} +\infty$

* Il existe $a > 1$ t.q. pour $x \geq a$

$$|P'(x)| \leq x^d \quad \text{donc pour } x \geq a$$

$$|P'(x)| \leq x^d < n x^{n-1}$$

$$P'(x) = \sum_{i=0}^{d-1} a_i' x^i$$

$$|P'(x)| \leq \sum_{i=0}^{d-1} |a_i'| x^i \leq x^{d-1} \sum_{i=0}^{d-1} |a_i'|$$

et P_n' ne s'annule pas sur $[a, +\infty[$

* On regarde $M = \| P'_{|[1,a]} \|_\infty$

Pour $n \geq \max(d+2, M+1)$, et $x \in [1, a]$

$$|P'(x)| \leq M < n x^{n-1} \quad (\text{car } x \geq 1)$$

donc pour $n \geq n_0 = \max(d+2, M+1)$, $P_n' > 0$

sur $[1, +\infty[$ car $P_n \xrightarrow[+\infty]{} +\infty$

Par TVI, pour $n \geq n_0$

$$\exists!\, x_n \in]1, +\infty[\text{ t.q. } P_n(x_n) = 0$$

b) Il existe $l > 1$ t.q. pour $x > l$, $|P(x)| \leq x^{d+1}$

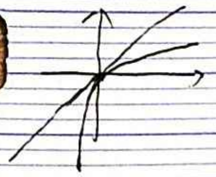
pour $n \geq n_0 + 1$ $\quad |P(x)| \leq x^{d+1} < x^n$ pour $x > 1$

donc $x_n \in [1, l]$ $\quad M' = \| P_{|[1,l]} \|_\infty$

$$x_n^n = P(x_n)$$

$$1 \leq x_n = (P(x_n))^{1/n} \leq M'^{1/n}$$ donc par encadrement, $x_n \to 1$.

1.4



a) $a_n - \ln(1+a_n)$

($\Leftarrow$) clair

($\Rightarrow$) on prend la fonction réciproque

$e^{x+y} = e^x e^y$

b) ($\Rightarrow$) $\frac{a_n}{\sqrt{n}} \to 0 \Rightarrow \frac{a_n}{n} \to 0$

$$\cancel{n\ln\left(1+\frac{a_n}{n}\right) = n\left(\frac{a_n}{n} + o\left(\frac{a_n}{n}\right)\right) = a_n + o(a_n)}$$

$$\cancel{\left(1+\frac{a_n}{n}\right)^n = e^{a_n+o(a_n)} = e^{a_n}(1+o(a_n))}$$

$\frac{a_n}{\sqrt{n}} \to 0$

$$\left(1+\frac{a_n}{n}\right)^n = e^{n\ln\left(1+\frac{a_n}{n}\right)} = e^{a_n} e^{-\frac{a_n^2}{2n} + o\left(\frac{a_n^2}{n}\right)} = e^{a_n} e^{o(1)} \sim e^{a_n}$$

$$\left(1+\frac{a_n}{n}\right)^n \sim e^{a_n}$$

($\Leftarrow$) On a : $e^{a_n} \sim \left(1+\frac{a_n}{n}\right)^n$

$u_n \to 1$

$\sqrt[n]{u_n} \to 1$

$$\Leftrightarrow \frac{e^{a_n}}{\left(1+\frac{a_n}{n}\right)^n} \to 1 \Rightarrow \frac{e^{\frac{a_n}{n}}}{1+\frac{a_n}{n}} \to 1$$

$$\Rightarrow \frac{a_n}{n} - \ln\left(1+\frac{a_n}{n}\right) \to 0$$

$\frac{t}{1+t/n} \leq t$

(Par a)) $\Rightarrow \frac{a_n}{n} \to 0$

$$e^{a_n - n\ln\left(1+\frac{a_n}{n}\right)} \to 1$$

$$= e^{a_n - n\left(\frac{a_n}{n} - \frac{a_n^2}{2n^2} + \alpha\left(\frac{a_n}{n}\right)\frac{a_n^3}{n^3}\right)} \quad , \alpha \text{ bornée}$$

$$= e^{\frac{a_n^2}{2n} + \underbrace{\alpha\left(\frac{a_n}{n}\right)}_{\text{borné}}\left(\frac{a_n^3}{n^2}\right)} \qquad o\left(\frac{a_n^2}{n}\right)$$

$$\frac{a_n^3}{n^2} \Big/ \frac{a_n^2}{2n} = \frac{a_n}{2n} \to 0$$

$$\frac{a_n^2}{2n} + \underbrace{o\left(\frac{a_n^2}{n}\right)}_{n\to+\infty} \to 0 \qquad \frac{a_n}{\sqrt{n}} \to 0$$

2.1

Par compacité de $[0,2\pi]$ et continuité de f,
Soit α t.q. $|f(\alpha)| = \inf_{t\in[0,2\pi]} |f(t)|$

$$f(x) = \int_\alpha^x f'(t)\,dt + f(\alpha)$$

$$|f(x)| \leq \int_\alpha^x |f'(t)|\,dt + |f(\alpha)|$$

$$\leq \int_0^{2\pi} |f'(t)|\,dt + |f(\alpha)|$$

Or $\left(\int_0^{2\pi} |f(t)|^2\,dt\right)^{1/2} \geq \left(|f(\alpha)|^2\, 2\pi\right)^{1/2} = |f(\alpha)|\sqrt{2\pi}$

Donc, $|f(x)| \leq \int_0^{2\pi} |f'(t)|\,dt + \frac{1}{\sqrt{2\pi}}\left(\int_0^{2\pi} |f(t)|^2\,dt\right)^{1/2}$

2.3

a)

$$\int_0^\pi x\, f(\sin x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{\pi} x f(\sin x)\,dx$$

$\sin x = \sin(\pi - x)$

$x = \pi - t$

$t = \pi - x$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_{\frac{\pi}{2}}^{0} (\pi - t) f(\sin(\pi - t))(-dt)$$

$$= \int_0^{\frac{\pi}{2}} x f(\sin x)\,dx + \int_0^{\frac{\pi}{2}} (\pi - t) f(\sin(t))\,dt$$

$$= \int_0^{\frac{\pi}{2}} \pi f(\sin x)\,dx$$

$$= \frac{\pi}{2} \times 2 \int_0^{\frac{\pi}{2}} f(\sin x)\,dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\,dx$$

b)

$$I = \int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

$$f : t \longmapsto \frac{t^{2n}}{t^{2n} + (1-t^2)^n}$$

par périodicité : $\int_0^{\pi} f(\sin x)\,dx = \int_0^{\pi} f(\cos x)\,dx$

$$\int_0^{\pi} f(\sin x) + f(\cos x)\,dx = \pi$$

$$I = \frac{\pi}{2} \times \frac{\pi}{2} = \frac{\pi^2}{4}$$

3.1 $\int_0^X \arctan(1+t)\,dt$

$$= \left[t \arctan(1+t)\right]_0^X - \int_0^X \frac{t}{1+(1+t)^2}\,dt$$

$$= X\arctan(1+X) - \int_0^X \frac{t+1}{1+(1+t)^2} - \frac{1}{1+(1+t)^2}\,dt$$

$$= X\arctan(1+X) - \frac{1}{2}\left[\ln(1+(1+t)^2)\right]_0^X + \left[\arctan(1+t)\right]_0^X$$

$$= X\arctan(1+X) - \frac{1}{2}\ln\left(1+(1+X)^2\right) - \frac{1}{2}\ln 2 + \arctan(1+X) - \frac{\pi}{4}$$

$\frac{du}{2\cancel{1+}u}$

$$\int_0^X \arctan t\,dt$$

$$= \left[t\arctan t\right]_0^X - \int_0^X \frac{t}{1+t^2}\,dt$$

$$= X\arctan X - \frac{1}{2}\left[\ln(t^2+1)\right]_0^X$$

$$= X\arctan X - \frac{1}{2}\ln(X^2+1)$$

$\text{arctg}\,\frac{1}{1+(1+X)X}$

$$\int_0^X \text{Arctg}(1+t) - \text{Arctg}(t)\,dt$$

$$= -\frac{1}{2}\ln 2 - \frac{\pi}{4} + X\left(\arctan(1+X) - \arctan X\right) - \frac{1}{2}\ln\left(\frac{1+(1+X)^2}{1+X^2}\right)$$

$$+ \arctan(1+X)$$

$\ln\left(1+\frac{2X+1}{1+X^2}\right)$

$$= -\frac{1}{2}\ln 2 - \frac{\pi}{4} + \underbrace{X\,\text{arctg}\frac{1}{1+X(X+1)}}_{O\left(\frac{1}{X}\right)\to 0} - \underbrace{\frac{1}{2}\ln\left(\frac{1+(1+X)^2}{1+X^2}\right)}_{\sim\frac{2X+1}{1+X^2}\to 0} + \underbrace{\text{arctg}(1+X)}_{\to\frac{\pi}{2}}$$

$$\boxed{=\frac{\pi}{4} - \frac{1}{2}\ln 2}$$

$$\int \frac{dx}{x-(a+ih)} = \frac{1}{2}\log\left((x-a)^2+h^2\right) + i\,\mathrm{arctg}\left(\frac{x-a}{h}\right)$$

$\log\left(1+\frac{1}{y^2}\right) \sim \frac{1}{y^2}$ en $+\infty$

$\frac{1}{y^2}$ est intégrable en $+\infty$

$\log\left(1+\frac{1}{y^2}\right) \sim -2\log y$ en 0.

$$\int_{x\to 0^+}^{1} \log t\,dt = \underbrace{\left[t\log t\right]_{\to 0}^{1}}_{\to 0} - \underbrace{\int_0^1 1\,dt}_{1}$$

$\log\left(1+\frac{1}{y^2}\right)$ intégrable en 0.

$$\int_0^{+\infty} \log\left(1+\frac{1}{y^2}\right)dy = \left[y\log\left(1+\frac{1}{y^2}\right)\right]_0^{+\infty} - \int_0^{+\infty} \frac{y\times\frac{-2}{y^3}}{1+\frac{1}{y^2}}\,dy$$

$$= \left[y\log\left(1+\frac{1}{y^2}\right)\right]_0^{+\infty} + 2\int_0^{+\infty}\frac{1}{y^2+1}\,dy$$

$$\lim_{X\to+\infty} X\log\left(1+\frac{1}{X^2}\right) - \lim_{X\to 0} X\log\left(1+\frac{1}{X^2}\right) + 2\left[\mathrm{arctg}\, y\right]_0^{+\infty}$$

$\sim X\times\frac{1}{X^2} \to 0$

$\sim -2X\log X \underset{0}{\to} 0$

$$= \qquad 2\times\frac{\pi}{2} = \pi$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan t}\,dt$$

$\sqrt{\tan t} = \frac{\sqrt{\sin t}}{\sqrt{\cos t}} \underset{\frac{\pi}{2}}{\sim} \frac{1}{\sqrt{\frac{\pi}{2}-t}}$ intégrable avec Riemann

$\sqrt{\tan t} = u$

$\tan t = u^2$

$(1+\tan^2 t)\,dt = 2u\,du$

$dt = \frac{2u}{1+u^4}\,du$

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan t}\,dt = \; = \int_0^{+\infty}\frac{2u^2}{1+u^4}\,du$$

$$\int_0^{+\infty}\frac{u^2+1}{1+u^4}\,du - \int_0^{+\infty}\frac{1}{1+u^4}\,du.$$

$$= \int_0^{+\infty}\frac{u^2+1}{(u^2+\sqrt{2}u+1)(u^2-\sqrt{2}u+1)}\,du - \int_0^{+\infty}\frac{1}{1+u^4}\,du$$

$$= \int_0^{+\infty}\frac{\frac{1}{2}}{u^2+\sqrt{2}u+1} + \frac{\frac{1}{2}}{u^2-\sqrt{2}u+1} - \frac{1}{1+u^4}\,du$$

$u = \frac{1}{v}$

$du = -\frac{1}{v^2}\,dv$

$$\int_0^{+\infty}\frac{1}{1+u^4}\,du$$

$$= \int_0^{+\infty}\frac{1}{1+\frac{1}{v^4}}\times\left(-\frac{1}{v^2}\right)dv$$

$$= -\int_0^{+\infty}\frac{v^2}{1+v^4}\,dv$$

3.2

$$h\sum_{n=1}^{+\infty} n^{\pi} e^{-nh}$$

$f(x) = x^{\pi} e^{-x}$, $\pi > 0$

$f'(x) = \pi x^{\pi-1} e^{-x} - x^{\pi} e^{-x} = (\pi - x) x^{\pi-1} e^{-x}$

$f' < 0$ pour $x > \pi$

$h > 0$ petit, $N = \left[\frac{\pi+1}{h}\right]$ $\qquad Nh \to \pi + 1$

$N \to +\infty$ avec $\frac{1}{h}$

f décroît sur $[Nh, +\infty[$

$\boxed{k \geq N}$ $\quad h f((k+1)h) \leq \int_{kh}^{(k+1)h} f \leq h \cdot f(kh)$ par $\downarrow$

De là, $\sum_{k=N+1}^{+\infty} h (k^{\pi} h^{\pi}) e^{-kh} \leq \int_{Nh}^{+\infty} x^{\pi} e^{-x} dx \leq \sum_{k=N}^{+\infty} h (kh)^{\pi} e^{-kh}$

Différence de somme $h \underbrace{(Nh)^{\pi} e^{-Nh}}_{\text{borné}} \xrightarrow[h \to 0]{} 0^+$

Puis $\sum_{n=1}^{N} h (nh)^{\pi} e^{-nh} \to \int_{0}^{\pi+1} x^{\pi} e^{-x} dx$

Somme de Riemann attachée à $0, \dots, Nh, \pi+1$

Ainsi, $h \sum_{n=1}^{+\infty} (kh)^{\pi} e^{-nh} \to \int_{0}^{+\infty} x^{\pi} e^{-x} dx = \Gamma(\pi+1)$

$$S(h) \underset{0^+}{\sim} \frac{\Gamma(\pi+1)}{h^{\pi+1}}$$

3.3. ▷ f définie sur $]0,1[\cup]1,+\infty[$

on pose $x = 1+h$ $(h > 0)$

En 1, $\int_{x}^{x^2} \frac{dt}{\log t} = \int_{1+h}^{1+2h+h^2} \frac{dt}{\log t} = \int_{h}^{h^2+2h} \frac{dt}{\log(1+t)}$

$= \int_{h}^{h^2+2h} \frac{dt}{t + \underbrace{\alpha(t)}_{\text{bornée}} t^2} = \int_{h}^{h^2+2h} \frac{1}{t} \left(1 - \underbrace{\beta(t)}_{\text{bornée}} t\right) dt$

$$= \underbrace{\int_h^{h^2+2h} \frac{dt}{t}}_{\log\left(\frac{h^2+2h}{h}\right) \to \log 2} - \underbrace{\int_h^{h^2+2h} \beta(t)\,dt}_{O(h) \to 0}$$

En 0^+, $\displaystyle\int_x^{x^2} \frac{dt}{\underbrace{\log t}_{\to 0}} \to 0$

b) $\displaystyle\left(\int_x^{x^2} \frac{dt}{\log t}\right)' = \left(\int^{x^2}\right)' - \left(\int^x\right)' = \frac{2x}{2\log x} - \frac{1}{\log x} = \frac{x-1}{\log x}$

$$\lim_{\varepsilon \to 0} \int_\varepsilon^{1-\varepsilon} \frac{t-1}{\log t}\,dt = \lim_{\varepsilon \to 0} f(1-\varepsilon) - f(\varepsilon) = \ln 2$$

3.4. $\displaystyle\int^{+\infty} \frac{2}{x}\,dx$ DV et la fonction est ≥ 0

$\displaystyle\log\left(\frac{f(x)}{f(0)}\right) = \int_0^x \frac{f'(t)}{f(t)}\,dt \underset{+\infty}{\sim} 2\log x$, $\log f(x) \sim \log x$

$f(x) \to +\infty$

$\displaystyle\int^{+\infty} f$ DV

Par X : $\underbrace{x f'(x)}_{\to \infty} \underset{+\infty}{\sim} \underbrace{2 f(x)}_{\to \infty}$

$$\int_0^x t f'(t)\,dt = \int_0^x 2 f(t)\,dt + o\left(\int_0^x f\right)$$

(IPP) $\displaystyle x f(x) - \int_0^x f(t)\,dt = 2\int_0^x f + o\left(\int_0^x f\right)$

$$x f(x) = 3\int_0^x f + o\left(\int_0^x f\right)$$

$\displaystyle\int_0^x f$ DV, donc $\displaystyle x f(x) \sim 3\int_0^x f$

$$\int_0^x f \sim \frac{x f(x)}{3}$$

1.1.b.

$$\left(1+\frac{1}{\sqrt{x}}\right)^x = e^{x\ln\left(1+\frac{1}{\sqrt{x}}\right)}$$

$$= e^{x\left(\frac{1}{\sqrt{x}} - \frac{1}{2x} + \frac{1}{3x^{3/2}} - \frac{1}{4x^2} + \frac{1}{5x^{5/2}} + o\left(\frac{1}{x^3}\right)\right)}$$

$$= e^{\sqrt{x} - \frac{1}{2} + \frac{1}{3\sqrt{x}} - \frac{1}{4x} + \frac{1}{5x^{3/2}} + o\left(\frac{1}{x^2}\right)}$$

$$= e^{\sqrt{x}-\frac{1}{2}}\, e^{\frac{1}{3\sqrt{x}} - \frac{1}{4x} + \frac{1}{5x^{3/2}} + o\left(\frac{1}{x^2}\right)}$$

$$= e^{\sqrt{x}-\frac{1}{2}}\left(1 + \frac{1}{3\sqrt{x}} - \frac{1}{4x} + \frac{1}{5x^{3/2}} + o\left(\frac{1}{x^2}\right)\right.$$

$$+ \frac{1}{2}\left(\frac{1}{9x} + \cancel{\frac{1}{16x^2}} - \frac{1}{6x^{3/2}} + o\left(\frac{1}{x^2}\right)\right)$$

$$\left. + \frac{1}{6}\left(\frac{1}{27x^{3/2}} + o\left(\frac{1}{x^2}\right)\right)\right)$$

$$\frac{1}{5} - \frac{1}{12} + \frac{1}{162}$$

$$= \_\_\_$$

$$= e^{\sqrt{x}-\frac{1}{2}}\left(1 + \frac{1}{3\sqrt{x}} - \frac{7}{36x} + \cancel{\frac{\quad}{x^{3/2}}} + o\left(\frac{1}{x^{3/2}}\right)\right)$$

3.1.(3)

$$u = \sqrt{\tan t}$$
$$u^2 = \tan t$$
$$2u\,du = (1+\tan^2 t)\,dt$$
$$dt = \frac{2u\,du}{1+u^4}$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan t}\, dt$$

$$= \int_0^{+\infty} \frac{2u^2}{1+u^4}\,du$$

$$= \int_0^{+\infty} \frac{\alpha_1 u + \beta_1}{u^2+\sqrt{2}u+1} + \frac{\alpha_2 u + \beta_2}{u^2-\sqrt{2}u+1}\,du$$

$$\alpha_1 + \alpha_2 = 0$$
$$\beta_1 + \beta_2 = 0$$
$$2 = -\sqrt{2}\alpha_1 + \beta_1 + \beta_2 + \sqrt{2}\alpha_2$$
$$0 = \alpha_1 - \cancel{2}\sqrt{2}\beta_1 + \sqrt{2}\beta_2 + \alpha_2$$
$$\beta_1 = \beta_2 = 0$$
$$\alpha_1 = -\frac{1}{\sqrt{2}}$$
$$\alpha_2 = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\int_0^{+\infty} \frac{-u}{u^2+\sqrt{2}u+1} + \frac{u}{u^2-\sqrt{2}u+1}\,du$$

$$= \frac{1}{2}\int_0^{+\infty} \frac{-\sqrt{2}u-1}{u^2+\sqrt{2}u+1} + \frac{1}{u^2+\sqrt{2}u+1} + \frac{\sqrt{2}u-1}{u^2-\sqrt{2}u+1} + \frac{1}{u^2-\sqrt{2}u+1}\,du$$

$$= \frac{1}{2}\left(-\frac{1}{\sqrt{2}}\Big[\ln(u^2+\sqrt{2}u+1)\Big]_0^{+\infty} + \frac{1}{\sqrt{2}}\Big[\ln(u^2-\sqrt{2}u+1)\Big]_0^{+\infty}\right.$$

$$\left. + \sqrt{2}\Big[\arctan(\sqrt{2}u+1)\Big]_0^{+\infty} + \sqrt{2}\Big[\arctan(\sqrt{2}u-1)\Big]_0^{+\infty}\right]$$

$$= \frac{1}{2}\left(\frac{1}{2}\left[\ln\frac{u^2-\sqrt{2}u+1}{u^2+\sqrt{2}u+1}\right]_0^{+\infty} + \sqrt{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) + \sqrt{2}\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\right)$$

$$= \frac{\sqrt{2}}{2}\pi$$

$$\frac{1}{\left(u+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}}\,du$$

$$= \frac{\sqrt{2}}{(\sqrt{2}u+1)^2+1}\,d(\sqrt{2}u)$$

2.3.b.

a) $2\int_0^\pi x f(\sin x)\,dx = \int_0^\pi x f(\sin x)\,dx + \int_0^\pi x f(\sin x)\,dx$

$= \int_0^\pi x f(\sin x)\,dx + \int_\pi^0 (\pi - t) f(\sin t)(-dt)$

$= \int_0^\pi \pi f(\sin x)\,dx = \pi \int_0^\pi f(\sin x)\,dx$

b) $f(\sin x) = \dfrac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}$ $\qquad f : t \longmapsto \dfrac{t^{2n}}{t^{2n} + (1-t^2)^n}$

MQ: $\displaystyle\int_0^\pi \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = \int_0^\pi \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

$\sin x = \cos\left(\frac{\pi}{2} - x\right)$

$\displaystyle\int_0^\pi \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

$t = \frac{\pi}{2} - x$

$dt = -dx$

$\displaystyle= \int_0^\pi \frac{\cos^{2n}\left(\frac{\pi}{2} - x\right)}{\sin^{2n} x + \cos^{2n} x}\,dx = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\cos^{2n} t}{\cos^{2n} t + \sin^{2n} t}(-dt)$

$\displaystyle= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^{2n} t}{\cos^{2n} t + \sin^{2n} t}\,dt = \int_0^\pi \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$

$\displaystyle\int_0^\pi \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = \frac{1}{2}\pi$

$\displaystyle\int_0^\pi \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} = \frac{\pi^2}{4}$

3.3. a.

<u>en $+\infty$:</u> $\displaystyle\int_x^{x^2} \frac{dt}{\ln t}$

$\displaystyle= \left[\frac{t}{\ln t}\right]_x^{x^2} + \underbrace{\int_x^{x^2} \frac{1}{(\ln t)^2}\,dt}_{\leq \frac{x^2}{(\ln x)^2} = o\left(\frac{x^2}{\ln x}\right)}$

$\displaystyle= \frac{x^2}{2\ln x} - \underbrace{\frac{x}{\ln x}}_{o\left(\frac{x^2}{\ln x}\right)} + o\left(\frac{x^2}{\ln x}\right) = \frac{x^2}{2\ln x} + o\left(\frac{x^2}{\ln x}\right)$

Donc, $\displaystyle\int_x^{x^2} \frac{dt}{\ln t} \underset{+\infty}{\sim} \frac{x^2}{2\ln x}$